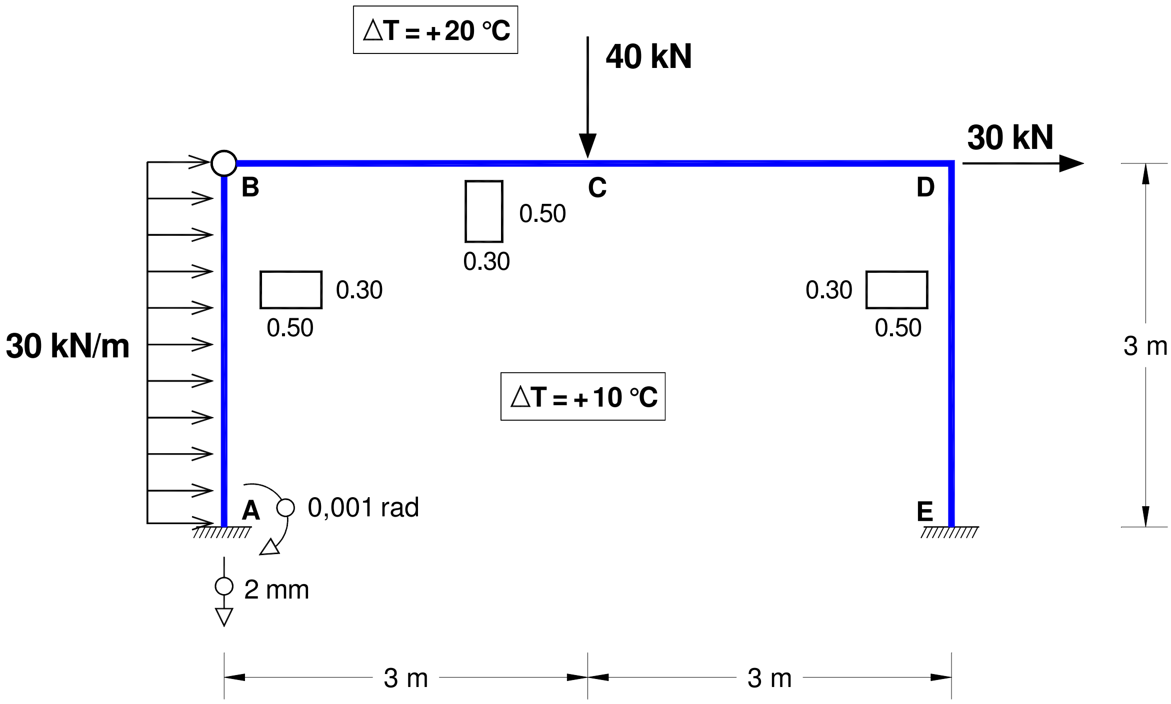
LICENCIATURA EM ENGENHARIA CIVIL

TEORIA DE ESTRUTURAS



MÉTODO DAS FORÇAS

ESTRUTURA CONTÍNUA

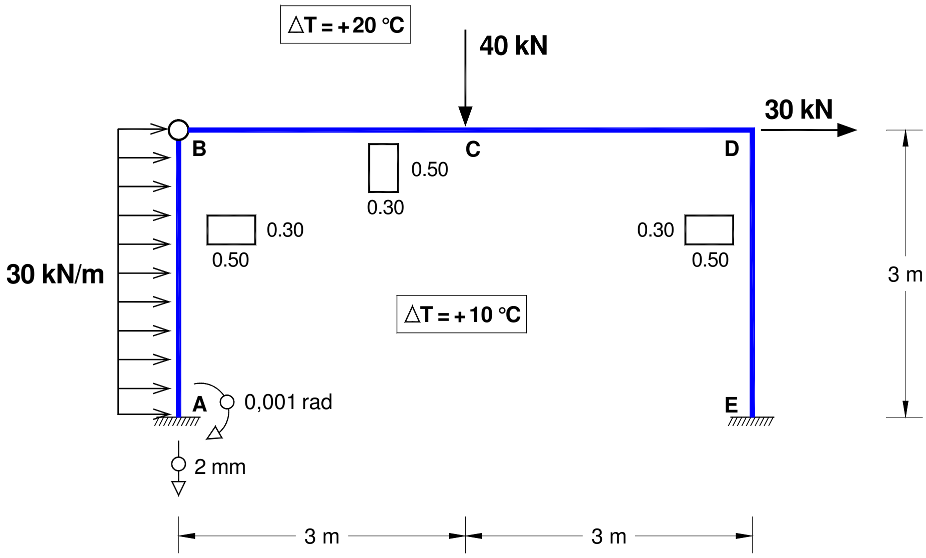
ISABEL ALVIM TELES

EXERCÍCIO

Considere a estrutura de betão (E=29 Gpa; α=10^{-5} /°C) constituída por dois pilares e uma viga com secções retangulares conforme representado na figura.

Para além do carregamento, a estrutura está submetida à seguinte variação de temperatura: +20 °C no exterior e +10 °C no interior.

O apoio **A** sofreu um assentamento de 2 mm e uma rotação de 0,001 rad conforme indicado na figura.



Na resolução deste exercício aplique o **Método das Forças** desprezando a deformabilidade por esforço transverso.

a) Determine as reações e trace os diagramas de esforço axial e momentos fletores da estrutura.

b) Determine o deslocamento horizontal da secção **D**.

c) Confirme os resultados obtidos nas alíneas anteriores utilizando o programa FTOOL.

RESOLUÇÃO

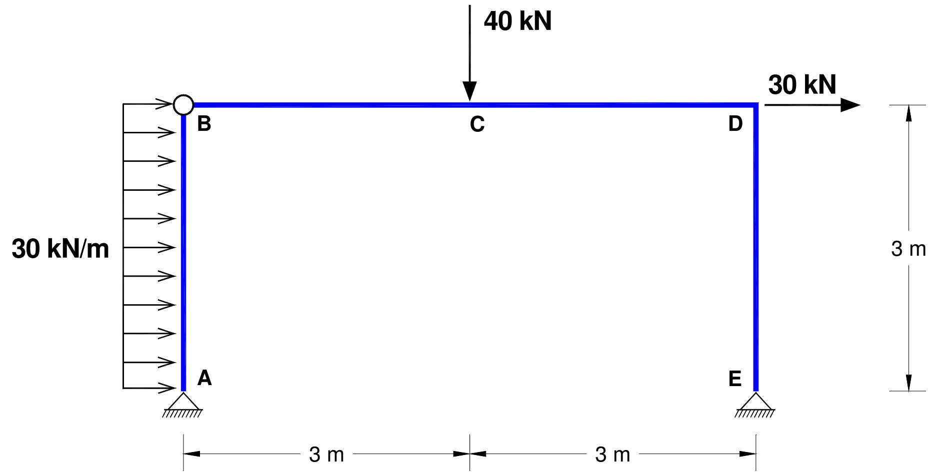
ALÍNEA a

A estrutura é constituída por dois corpos unidos entre si por uma rótula, o que configura um arco-de-3-rótulas. Para ser isostática teria de ter somente 4 incógnitas como reações. Como apresenta 6 reações (2 momentos de encastramento, 2 reações horizontais e 2 reações verticais), a estrutura é hiperestática de grau 2.

Vamos adoptar para sistema base (**S$_0$**) a estrutura com apoios duplos em **A** e **E**, configurando assim um arco-de-3-rótulas.

Na aplicação do **Método das Forças** as 2 incógnitas hiperstáticas corresponderão aos momentos de encastramento nos apoios **A** e **E**.

Sistema Base **(S_0)**

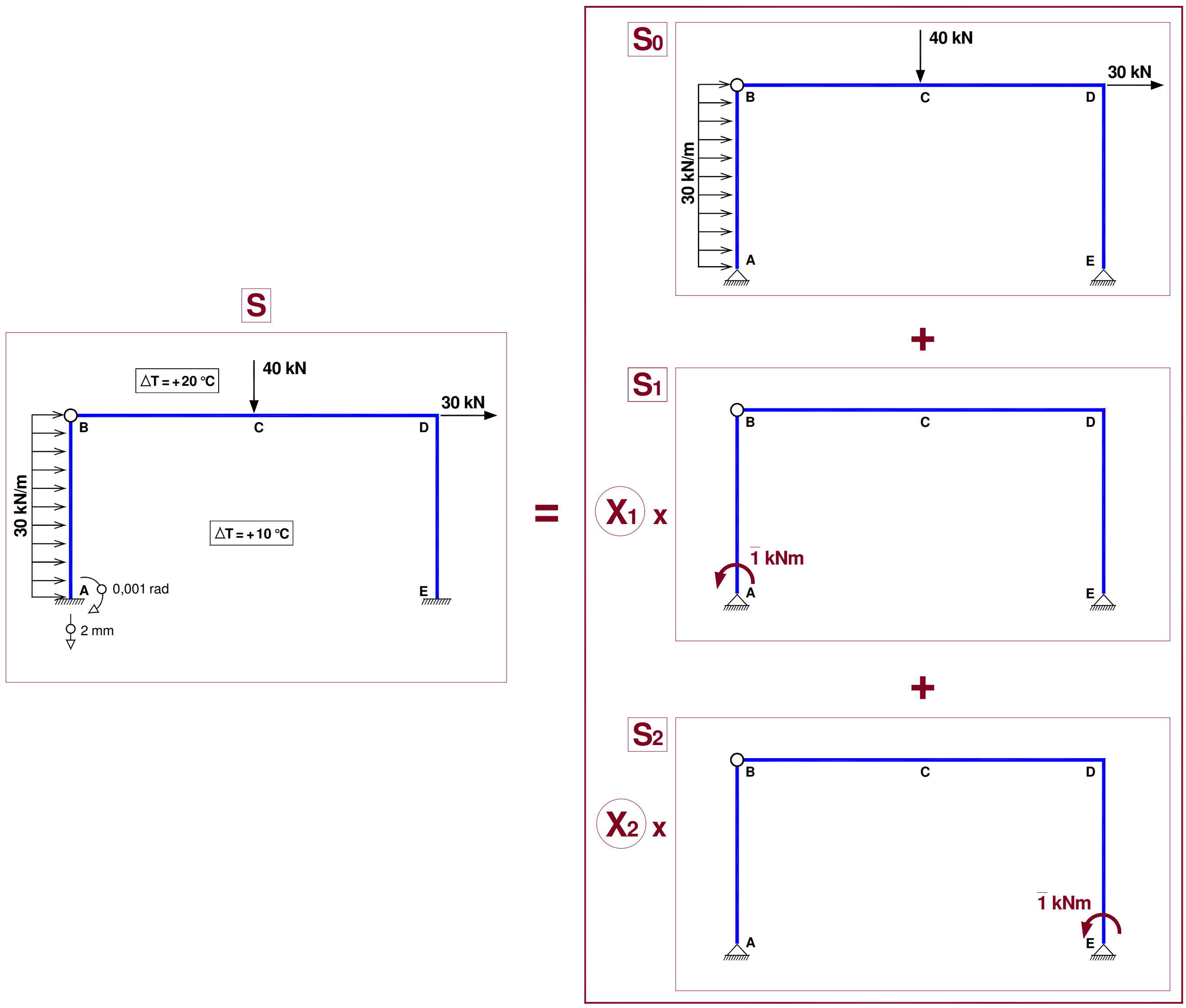


A estrutura real (**S**) vai ser decomposta nos sistemas **S_0**, **S_1** e **S_2** .

$$\boxed{S = S_0 + X_1 \times S_1 + X_2 \times S_2}$$

X_1 – incógnita hiperestática, correspondente ao momento de encastramento no apoio **A**.

X_2 – incógnita hiperestática, correspondente ao momento de encastramento no apoio **E**.

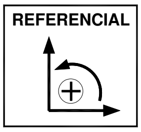


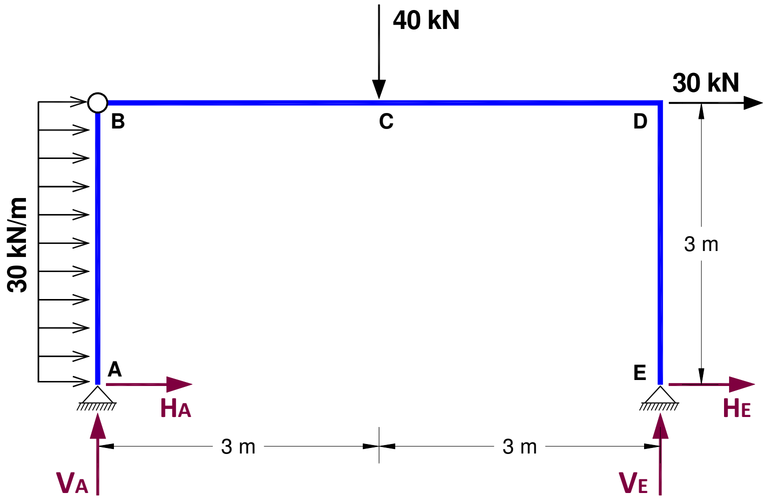
• Cálculo do sistema S_0

Equações de equilíbrio (arco de 3 rótulas)

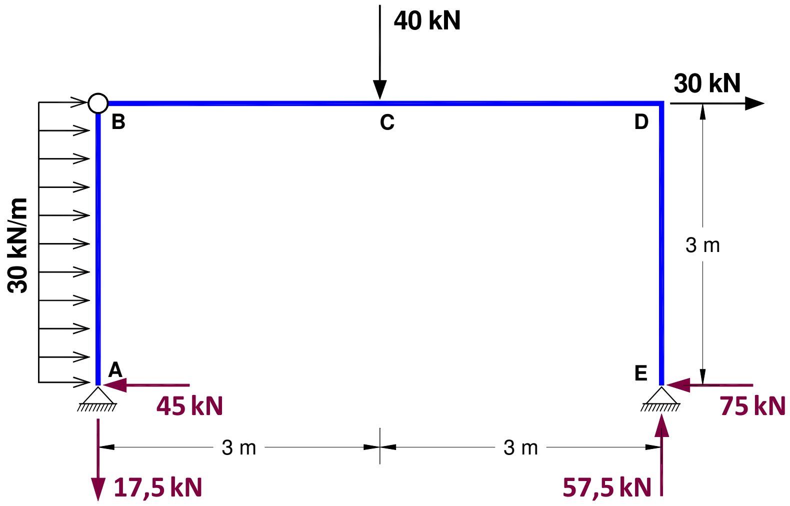
Toda a estrutura $\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \end{cases}$

Corpo $\overline{AB}$ $\quad \sum M^{AB.}_{rot.B} = 0$





$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \\ \sum M^{AB.}_{rot.B} = 0 \end{cases} \Rightarrow \begin{cases} H_A + H_E + 30\times 3 + 30 = 0 \\ V_A + V_E - 40 = 0 \\ -V_A\times 6 - 90\times 1{,}5 + 40\times 3 - 30\times 3 = 0 \\ H_A\times 3 + 90\times 1{,}5 = 0 \end{cases} \Rightarrow \begin{cases} H_A = -45\,kN \leftarrow \\ V_A = -17{,}5\ kN \downarrow \\ H_E = -75\,kN \leftarrow \\ V_E = 57{,}5\ kN \uparrow \end{cases}$$



Barra AB

$M(z) = 45z - 15z^2 \Rightarrow \begin{cases} M_A(z=0) = 0 \\ M_{máx}(z=1{,}5) = 33{,}75\ kNm \\ M_B(z=3) = 0 \end{cases}$

$V(z) = 45 - 30z$

$V = 0 \Rightarrow Z = 1{,}5\ m \Rightarrow M_{máx.}$

$N(z) = M_B(z=3) = 17{,}5\ kN$

Barra BC

$$M(z) = -17{,}5\,z \quad \Rightarrow \quad \begin{cases} M_{B\,(z=0)} = 0 \\ M_{C\,(z=3)} = -52{,}5 \text{ kNm} \end{cases}$$

$$N(z) = -45 \text{ kN}$$

Barra CD

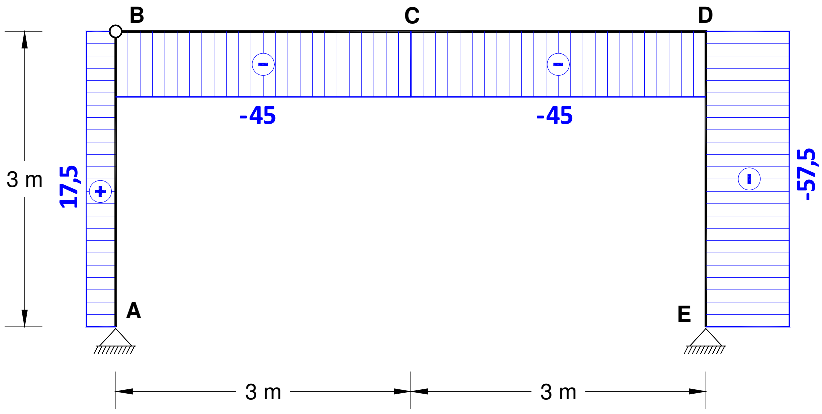
$$M(z) = -52{,}5 - 57{,}5\,z \quad \Rightarrow \quad \begin{cases} M_{C\,(z=0)} = -55{,}5 \text{ kNm} \\ M_{D\,(z=3)} = -225 \text{ kNm} \end{cases}$$

$$N(z) = -45 \text{ kN}$$

Barra ED

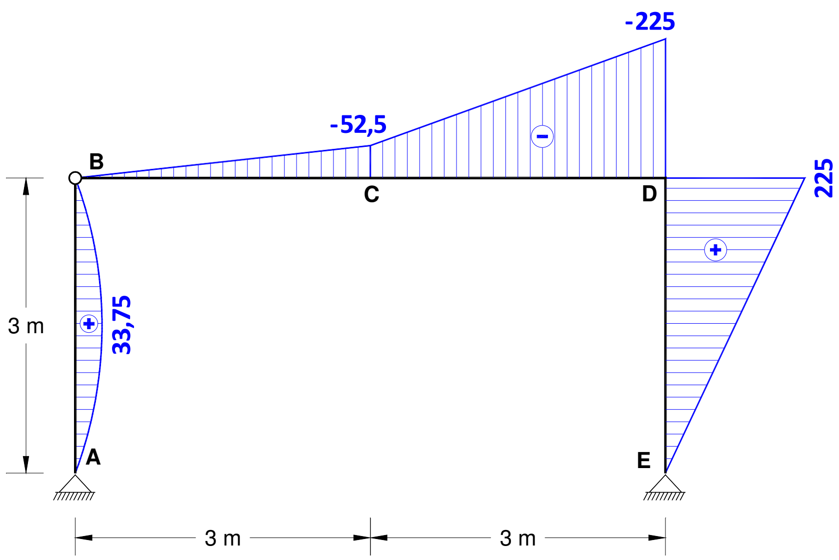
$$M(z) = 75\,z \quad \Rightarrow \quad \begin{cases} M_{E\,(z=0)} = 0 \\ M_{D\,(z=3)} = 225 \text{ kNm} \end{cases}$$

$$N(z) = -57{,}5 \text{ kN}$$



SISTEMA S_0

Esforço Axial (kN)



SISTEMA S_0

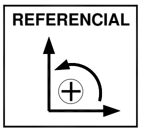
Momento Fletor (kNm)

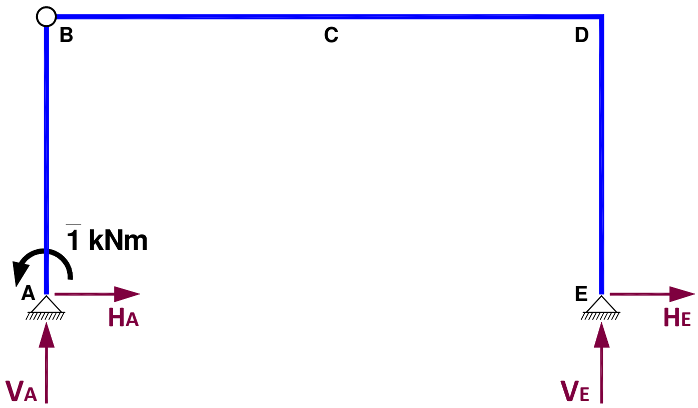
• Cálculo do sistema S_1

Equações de equilíbrio (arco de 3 rótulas)

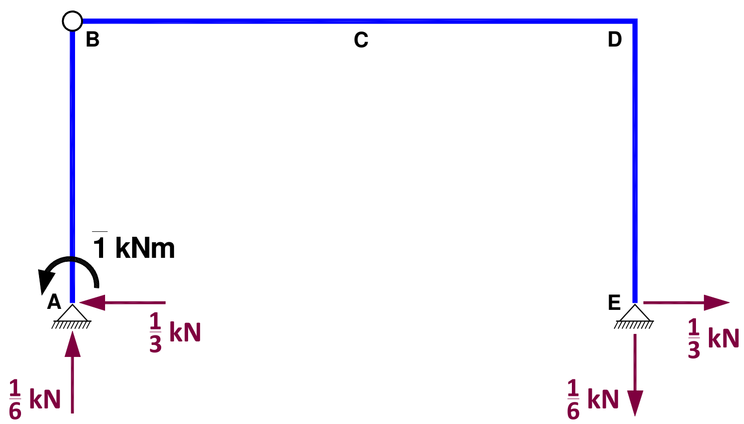
$$\text{Toda a estrutura} \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \end{cases}$$

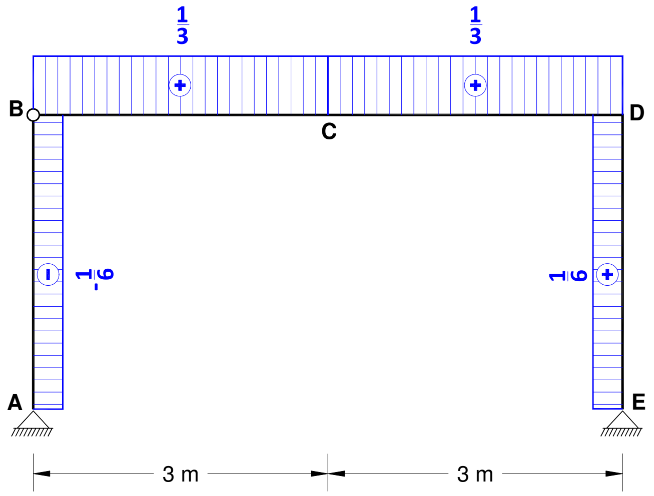
$$\text{Corpo } \overline{AB} \quad \sum M^{AB.}_{rot.B} = 0$$





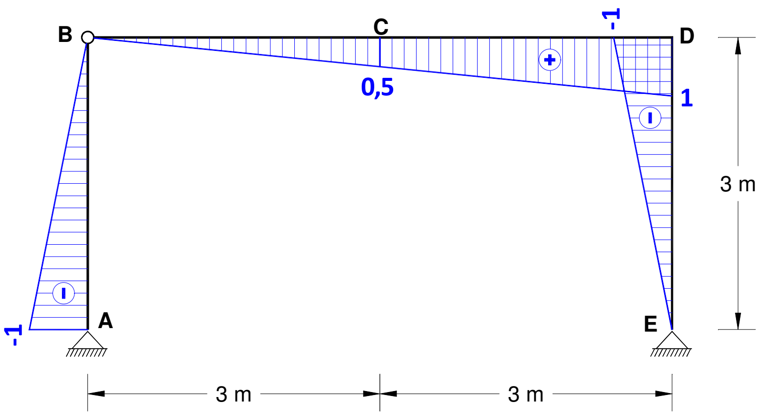
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \\ \sum M^{AB.}_{rot.B} = 0 \end{cases} \Rightarrow \begin{cases} H_A + H_E = 0 \\ V_A + V_E = 0 \\ -V_A \times 6 + 1 = 0 \\ H_A \times 3 + 1 = 0 \end{cases} \Rightarrow \begin{cases} H_A = -\frac{1}{3} kN \leftarrow \\ V_A = \frac{1}{6} kN \uparrow \\ H_E = \frac{1}{3} kN \rightarrow \\ V_E = -\frac{1}{6} kN \downarrow \end{cases}$$





SISTEMA S_1

<u>Esforços Axiais</u> (kN)



SISTEMA S_1

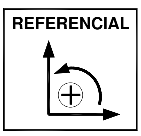
<u>Momentos Fletores</u> (kNm)

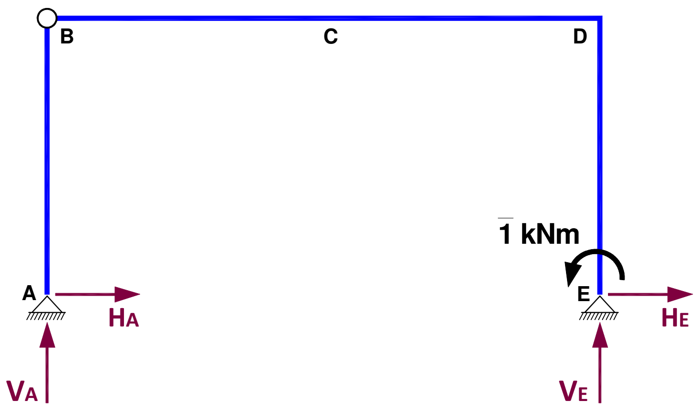
• <u>Cálculo do sistema S_2</u>

Equações de equilíbrio (arco de 3 rótulas)

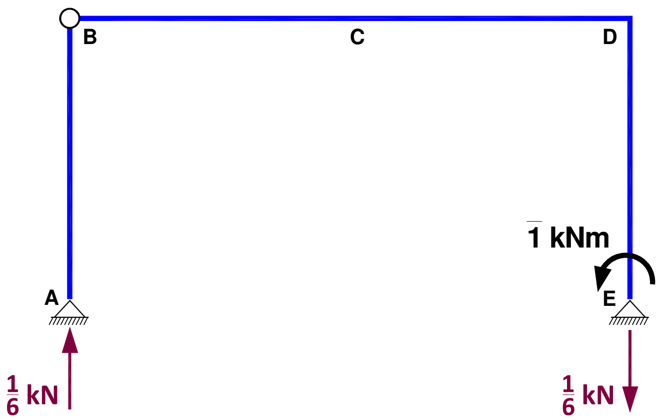
Toda a estrutura $\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \end{cases}$

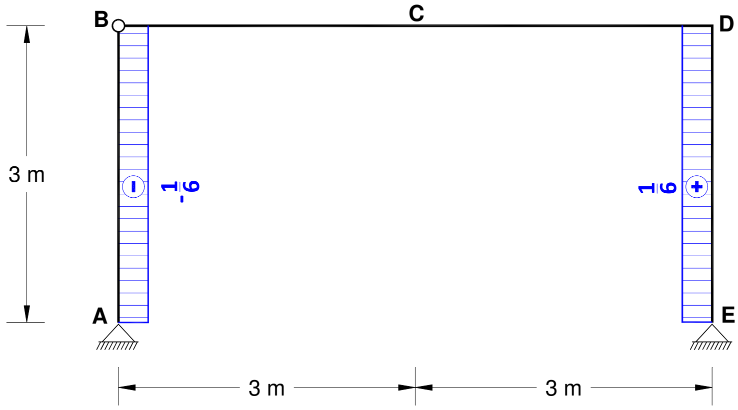
Corpo $\overline{AB}$ $\quad \sum M^{AB.}_{rot.B} = 0$





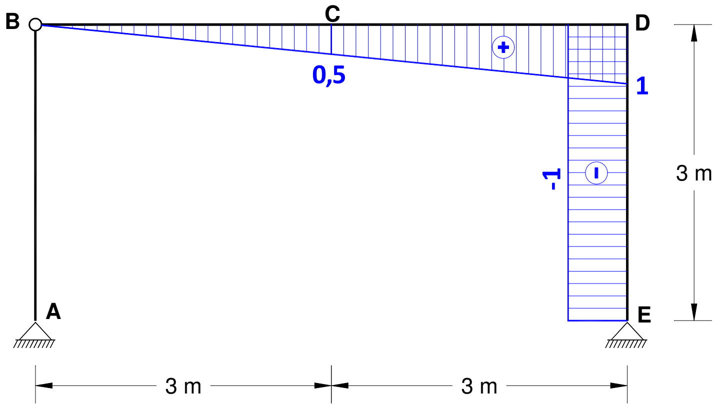
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \\ \sum M^{AB.}_{rot.B} = 0 \end{cases} \Rightarrow \begin{cases} H_A + H_E = 0 \\ V_A + V_E = 0 \\ -V_A \times 6 + 1 = 0 \\ H_A \times 3 = 0 \end{cases} \Rightarrow \begin{cases} H_A = 0 \\ V_A = \frac{1}{6}\ \text{kN} \uparrow \\ H_E = 0 \\ V_E = -\frac{1}{6}\ \text{kN} \downarrow \end{cases}$$





SISTEMA S_2

Esforços Axiais (kN)



SISTEMA S_2

Momentos Fletores (kNm)

• Determinação das incógnitas hiperestáticas

MÉTODO DAS FORÇAS

$$\begin{cases} \sum F_{ext\,S1} \times \Delta_R = \int \frac{N_1.N}{E\,A}\,dz + \int \frac{M_1.M}{E\,I}\,dz + \int N_1.\,\alpha\,.\,T_{méd}\,dz + \int M_1\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,dz \\ \sum F_{ext\,S2} \times \Delta_R = \int \frac{N_2.N}{E\,A}\,dz + \int \frac{M_2.M}{E\,I}\,dz + \int N_2.\,\alpha\,.\,T_{méd}\,dz + \int M_2\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,d \end{cases}$$

Considerando:

$$\begin{cases} N = N_0 + \mathbf{X_1}\,N_1 + \mathbf{X_2}\,N_2 \\ M = M_0 + \mathbf{X_1}\,M_1 + \mathbf{X_2}\,M_2 \end{cases} \Rightarrow$$

$$\begin{cases} \sum F_{ext\,S1} \times \Delta_R = \int \frac{N_1.N_0}{E\,A}dz + \mathbf{X_1}\int \frac{N_1.N_1}{E\,A}dz + \mathbf{X_2}\int \frac{N_1.N_2}{E\,A}dz + \int \frac{M_1.M_0}{E\,I}\,dz + \\ \qquad + \mathbf{X_1}\int \frac{M_1.M_1}{E\,I}\,dz + \mathbf{X_2}\int \frac{M_1.M_2}{E\,I}\,dz + \\ \qquad + \int N_1.\,\alpha\,.\,T_{méd}\,dz + \int M_1\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,dz \\ \\ \sum F_{ext\,S2} \times \Delta_R = \int \frac{N_2.N}{E\,A}\,dz + \mathbf{X_1}\int \frac{N_2.N_1}{E\,A}dz + \mathbf{X_2}\int \frac{N_2.N_2}{E\,A}dz + \int \frac{M_2.M_0}{E\,I}\,dz + \\ \qquad + \mathbf{X_1}\int \frac{M_2.M_1}{E\,I}\,dz + \mathbf{X_2}\int \frac{M_2.M_2}{E\,I}\,dz + \\ \qquad + \int N_2.\,\alpha\,.\,T_{méd}\,dz + \int M_2\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,dz \end{cases}$$

Sistema de equações para cálculo das incógnitas hiperstáticas:

$$\begin{cases} \delta_{10} + \delta_{11}\,.\,\mathbf{X_1} + \delta_{12}\,.\,\mathbf{X_2} = 0 \\ \delta_{20} + \delta_{21}\,.\,\mathbf{X_1} + \delta_{22}\,.\,\mathbf{X_2} = 0 \end{cases}$$

sendo:

$$\delta_{10} = \int \frac{N_1.N_0}{E\,A}\,dz + \int \frac{M_1.M_0}{E\,I}\,dz + \int N_1.\,\alpha\,.\,T_{méd}\,dz + \int M_1\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,dz - \sum F_{ext\,S1} \times \Delta_R$$

$$\delta_{11} = \int \frac{N_1.N_1}{E\,A}\,dz + \int \frac{M_1.M_1}{E\,I}\,dz$$

$$\delta_{12} = \int \frac{N_1.N_2}{E\,A}\,dz + \int \frac{M_1.M_2}{E\,I}\,dz$$

$$\delta_{20} = \int \frac{N_2.N_0}{E\,A}\,dz + \int \frac{M_2.M_0}{E\,I}\,dz + \int N_2.\,\alpha\,.\,T_{méd}\,dz + \int M_2\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,dz - \sum F_{ext\,S2} \times \Delta_R$$

$$\delta_{21} = \delta_{12} = \int \frac{N_2.N_1}{E\,A}\,dz + \int \frac{M_2.M_1}{E\,I}\,dz$$

$$\delta_{22} = \int \frac{N_2.N_2}{E\,A}\,dz + \int \frac{M_2.M_2}{E\,I}\,dz$$

Características das secções transversais de todas as barras	
Secção retangular $\begin{cases} b=0{,}30\,m \\ h=0{,}50\,m \end{cases}$	EA = 4,35 x 10^6 kPa.m^2 EI = 90 625 kPa.m^4 Coef. dilatação térmica: $\alpha = 10^{-5}$ /°C

Barra AB $\Rightarrow \int \frac{N_1.N_0}{EA}\,dz = \frac{1}{EA}\times\left(-\frac{1}{6}\right)\times 17{,}5\times 3 = -\frac{8{,}75}{EA} = -\frac{8{,}75}{4{,}35\times10^6}$ Barras BCD $\Rightarrow \int \frac{N_1.N_0}{EA}\,dz = \frac{1}{EA}\times\frac{1}{6}(-45)\times 6 = -\frac{90}{EA} = -\frac{90}{4{,}35\times10^6}$ Barra ED $\Rightarrow \int \frac{N_1.N_0}{EA}\,dz = \frac{1}{EA}\times\frac{1}{6}(-57{,}5)\times 3 = -\frac{28{,}75}{EA} = -\frac{28{,}75}{4{,}35\times10^6}$	$\int \frac{N_1.N_0}{EA}\,dz = -\frac{127{,}5}{4{,}35\times10^6}$

Barra AB $\Rightarrow \int \frac{M_1.M_0}{EI}\,dz = \frac{1}{EI}\times\frac{1}{3}(-1)\times 33{,}75\times 3 = -\frac{33{,}75}{EI} = -\frac{33{,}75}{90\,625}$ Barra BC $\Rightarrow \int \frac{M_1.M_0}{EI}\,dz = \frac{1}{EI}\times\frac{1}{3}\times 0{,}5\times(-52{,}5)\times 3 = -\frac{26{,}25}{EI} = -\frac{26{,}25}{90\,625}$ Barra CD $\Rightarrow \int \frac{M_1.M_0}{EI}\,dz =$ $= \frac{1}{EI}\times\frac{3}{6}\times\left(2\times0{,}5\times(-52{,}5)+2\times1\times(-225)+0{,}5\times(-225)+1\times(-52{,}5)\right) =$ $= -\frac{333{,}75}{EI} = -\frac{333{,}75}{90\,625}$ Barra ED $\Rightarrow \int \frac{M_1.M_0}{EI}\,dz = \frac{1}{EI}\times\frac{1}{3}\times(-1)\times 225\times 3 = -\frac{225}{EI} = -\frac{225}{90\,625}$	$\int \frac{M_1.M_0}{EI}\,dz = -\frac{618{,}75}{90\,625}$

Barra AB $\Rightarrow \int N_1.\,\alpha\,.\,T_{méd}\,dz = \left(-\frac{1}{6}\right)\times 3x10^{-5}\times\frac{20+10}{2} = -7{,}5x10^{-5}$ Barras BCD $\Rightarrow \int N_1.\,\alpha\,.\,T_{méd}\,dz = \frac{1}{3}\times 6x10^{-5}\times\frac{20+10}{2} = 30x10^{-5}$ Barra DE $\Rightarrow \int N_1.\,\alpha\,.\,T_{méd}\,dz = \frac{1}{6}\times 3x10^{-5}\times\frac{20+10}{2} = 7{,}5x10^{-5}$	$\int N_1.\,\alpha\,.\,T_{méd}\,dz = 30x10^{-5}$

Barra AB $\Rightarrow \int M_1\,.\,\alpha\,\frac{T_{inf}-T_{sup}}{h}\,dz = \frac{-1\times3}{2}\times10^{-5}\times\frac{10-20}{0{,}50} = 30x10^{-5}$ Barras BCD $\Rightarrow \int M_1\,.\,\alpha\,\frac{T_{inf}-T_{sup}}{h}\,dz = \frac{1\times6}{2}\times10^{-5}\times\frac{10-20}{0{,}50} = -60x10^{-5}$ Barra DE $\Rightarrow \int M_1\,.\,\alpha\,\frac{T_{inf}-T_{sup}}{h}\,dz = \frac{-1\times3}{2}\times10^{-5}\times\frac{20-10}{0{,}50} = -30x10^{-5}$	$\int M_1\,.\,\alpha\,\frac{T_{inf}-T_{sup}}{h}\,dz = -60x10^{-5}$

$$\sum F_{ext\,S1}\times\Delta_R = -\frac{1}{6}\times 0{,}002 - 1\times 0{,}001 = -\frac{400}{3}\times 10^{-5}$$

$$\delta_{10} = \int \frac{N_1.N_0}{E\,A}\,dz + \int \frac{M_1.M_0}{E\,I}\,dz + \int N_1.\,\alpha\,.\,T_{méd}\,dz + \int M_1\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,dz - \sum F_{ext\,S1} \times \Delta_R =$$

$$= -\frac{127{,}5}{4{,}35\times10^6} - \frac{618{,}75}{90\,625} + 30 \times 10^{-5} - 60 \times 10^{-5} + \frac{400}{3} \times 10^{-5} = -582{,}356 \times 10^{-5}$$

Barra AB $\Rightarrow \int \frac{N_1.N_1}{E\,A}dz = \frac{1}{E\,A}\times\left(-\frac{1}{6}\right)\times3\times\left(-\frac{1}{6}\right) = \frac{1}{E\,A}\times\frac{1}{12} = \frac{1}{12}\times\frac{1}{4{,}35\times10^6}$ Barras BCD $\Rightarrow \int \frac{N_1.N_1}{E\,A}dz = \frac{1}{E\,A}\times\frac{1}{3}\times6\times\frac{1}{3} = \frac{1}{E\,A}\times\frac{2}{3} = \frac{2}{3}\times\frac{1}{4{,}35\times10^6}$ Barra ED $\Rightarrow \int \frac{N_1.N_1}{E\,A}dz = \frac{1}{E\,A}\times\frac{1}{6}\times3\times\frac{1}{6} = \frac{1}{E\,A}\times\frac{1}{12} = \frac{1}{12}\times\frac{1}{4{,}35\times10^6}$	$\int \frac{N_1.N_1}{E\,A}dz = \frac{5}{6}\times\frac{1}{4{,}35\times10^6}$

Barra AB $\Rightarrow \int \frac{M_1.M_1}{E\,I}dz = \frac{1}{E\,I}\times\frac{(-1)\times(-1)\times3}{3} = \frac{1}{E\,I} = \frac{1}{90\,625}$ Barra BCD $\Rightarrow \int \frac{M_1.M_1}{E\,I}dz = \frac{1}{E\,I}\times\frac{1\times1\times6}{3} = \frac{2}{E\,I} = \frac{2}{90\,625}$ Barra ED $\Rightarrow \int \frac{M_1.M_1}{E\,I}dz = \frac{1}{E\,I}\times\frac{1\times1\times3}{3} = \frac{1}{E\,I} = \frac{1}{90\,625}$	$\int \frac{M_1.M_1}{E\,I}dz = \frac{4}{90\,625}$

$$\delta_{11} = \int \frac{N_1.N_1}{E\,A}\,dz + \int \frac{M_1.M_1}{E\,I}\,dz = \frac{5}{6}\times\frac{1}{4{,}35\times10^6} + \frac{4}{90\,625} = 4{,}43295 \times 10^{-5}$$

Barra AB $\Rightarrow \int \frac{N_1.N_2}{E\,A}dz = \frac{1}{E\,A}\times\left(-\frac{1}{6}\right)\times3\times\left(-\frac{1}{6}\right) = \frac{1}{E\,A}\times\frac{1}{12} = \frac{1}{12}\times\frac{1}{4{,}35\times10^6}$ Barras BCD $\Rightarrow \int \frac{N_1.N_2}{E\,A}dz = 0$ Barra ED $\Rightarrow \int \frac{N_1.N_2}{E\,A}dz = \frac{1}{E\,A}\times\frac{1}{6}\times3\times\frac{1}{6} = \frac{1}{E\,A}\times\frac{1}{12} = \frac{1}{12}\times\frac{1}{4{,}35\times10^6}$	$\int \frac{N_1.N_2}{E\,A}dz = \frac{1}{6}\times\frac{1}{4{,}35\times10^6}$

Barra AB $\Rightarrow \int \frac{M_1.M_2}{E\,I}dz = 0$ Barra BCD $\Rightarrow \int \frac{M_1.M_2}{E\,I}dz = \frac{1}{E\,I}\times\frac{1\times1\times6}{3} = \frac{2}{E\,I} = \frac{2}{90\,625}$ Barra ED $\Rightarrow \int \frac{M_1.M_2}{E\,I}dz = \frac{1}{E\,I}\times\frac{1\times1\times3}{2} = \frac{1{,}5}{E\,I} = \frac{1{,}5}{90\,625}$	$\int \frac{M_1.M_2}{E\,I}dz = \frac{3{,}5}{90\,625}$

$$\delta_{12} = \int \frac{N_1.N_2}{E\,A}\,dz + \int \frac{M_1.M_2}{E\,I}\,dz = \frac{1}{6}\times\frac{1}{4{,}35\times10^6} + \frac{3{,}5}{90\,625} = 3{,}86590 \times 10^{-5}$$

Barra AB $\Rightarrow \int \frac{N_2.N_0}{EA} dz = \frac{1}{EA} \times \left(-\frac{1}{6}\right) \times 17{,}5 \times 3 = -\frac{8{,}75}{EA} = -\frac{8{,}75}{4{,}35\times10^6}$ Barras BCD $\Rightarrow \int \frac{N_2.N_0}{EA} dz = 0$ Barra ED $\Rightarrow \int \frac{N_2.N_0}{EA} dz = \frac{1}{EA} \times \frac{1}{6} (-57{,}5) \times 3 = -\frac{28{,}75}{EA} = -\frac{28{,}75}{4{,}35\times10^6}$	$\int \frac{N_2.N_0}{EA} dz = -\frac{37{,}5}{4{,}35\times10^6}$

Barra AB $\Rightarrow \int \frac{M_2.M_0}{EI} dz = 0$ Barra BC $\Rightarrow \int \frac{M_2.M_0}{EI} dz = \frac{1}{EI} \times \frac{1}{3} \times 0{,}5 \times (-52{,}5) \times 3 = -\frac{26{,}25}{EI} = -\frac{26{,}25}{90\ 625}$ Barra CD $\Rightarrow \int \frac{M_2.M_0}{EI} dz =$ $= \frac{1}{EI} \times \frac{3}{6} \times \left(2\times0{,}5\times(-52{,}5)+2\times1\times(-225)+0{,}5\times(-225)+1\times(-52{,}5)\right) =$ $= -\frac{333{,}75}{EI} = -\frac{333{,}75}{90\ 625}$ Barra ED $\Rightarrow \int \frac{M_2.M_0}{EI} dz = \frac{1}{EI} \times \frac{1}{2} \times (-1) \times 225 \times 3 = -\frac{337{,}5}{EI} = -\frac{337{,}5}{90\ 625}$	$\int \frac{M_2.M_0}{EI} dz = -\frac{697{,}5}{90\ 625}$

Barra AB $\Rightarrow \int N_2.\alpha.T_{méd}\ dz = \left(-\frac{1}{6}\right) \times 3\text{x}10^{-5} \times \frac{20+10}{2} = -7{,}5\text{x}10^{-5}$ Barras BCD $\Rightarrow \int N_2.\alpha.T_{méd}\ dz = 0$ Barra DE $\Rightarrow \int N_2.\alpha.T_{méd}\ dz = \frac{1}{6} \times 3\text{x}10^{-5} \times \frac{20+10}{2} = 7{,}5\text{x}10^{-5}$	$\int N_2.\alpha.T_{méd}\ dz = 0$

Barra AB $\Rightarrow \int M_2.\alpha \frac{T_{inf} - T_{sup}}{h} dz = 0$ Barras BCD $\Rightarrow \int M_2.\alpha \frac{T_{inf} - T_{sup}}{h} dz = \frac{1\times6}{2} \times 10^{-5} \times \frac{10-20}{0{,}50} = -60\text{x}10^{-5}$ Barra DE $\Rightarrow \int M_2.\alpha \frac{T_{inf} - T_{sup}}{h} dz = -3\times10^{-5} \times \frac{20-10}{0{,}50} = -60\text{x}10^{-5}$	$\int M_2.\alpha \frac{T_{inf} - T_{sup}}{h} dz = -120\text{x}10^{-5}$

$$\sum F_{ext\ S2} \times \Delta_R = -\frac{1}{6} \times 0{,}002 = -\frac{100}{3} \times 10^{-5}$$

$$\delta_{20} = \int \frac{N_2.N_0}{EA} dz + \int \frac{M_2.M_0}{EI} dz + \int N_2.\alpha.T_{méd}\ dz + \int M_2.\alpha \frac{T_{inf} - T_{sup}}{h} dz - \sum F_{ext\ S2} \times \Delta_R =$$

$$= -\frac{37{,}5}{4{,}35\times10^6} - \frac{697{,}5}{90\ 625} + 0 - 120\text{x}10^{-5} + \frac{100}{3} \times 10^{-5} = -857{,}184 \times 10^{-5}$$

$$\delta_{21} = \int \frac{N_2.N_1}{E\,A}\,dz + \int \frac{M_2.M_1}{E\,I}\,dz = \delta_{12} = 3{,}86590 \times 10^{-5}$$

Barra AB $\Rightarrow \int \frac{N_2.N_2}{E\,A}\,dz = \frac{1}{E\,A} \times \left(-\frac{1}{6}\right) \times 3 \times \left(-\frac{1}{6}\right) = \frac{1}{E\,A} \times \frac{1}{12} = \frac{1}{12} \times \frac{1}{4{,}35 \times 10^6}$ Barras BCD $\Rightarrow \int \frac{N_2.N_2}{E\,A}\,dz = 0$ Barra ED $\Rightarrow \int \frac{N_2.N_2}{E\,A}\,dz = \frac{1}{E\,A} \times \frac{1}{6} \times 3 \times \frac{1}{6} = \frac{1}{E\,A} \times \frac{1}{12} = \frac{1}{12} \times \frac{1}{4{,}35 \times 10^6}$	$\int \frac{N_2.N_2}{E\,A}\,dz = \frac{1}{6} \times \frac{1}{4{,}35 \times 10^6}$

Barra AB $\Rightarrow \int \frac{M_2.M_2}{E\,I}\,dz = 0$ Barra BCD $\Rightarrow \int \frac{M_2.M_2}{E\,I}\,dz = \frac{1}{E\,I} \times \frac{1 \times 1 \times 6}{3} = \frac{2}{E\,I} = \frac{2}{90\,625}$ Barra ED $\Rightarrow \int \frac{M_2.M_2}{E\,I}\,dz = \frac{1}{E\,I} \times (-1) \times (-1) \times 3 = \frac{3}{E\,I} = \frac{3}{90\,625}$	$\int \frac{M_2.M_2}{E\,I}\,dz = \frac{5}{90\,625}$

$$\delta_{22} = \int \frac{N_2.N_2}{E\,A}\,dz + \int \frac{M_2.M_2}{E\,I}\,dz = \frac{1}{6} \times \frac{1}{4{,}35 \times 10^6} + \frac{5}{90\,625} = 5{,}52107 \times 10^{-5}$$

$$\begin{cases} \delta_{10} + \delta_{11} . \mathbf{X_1} + \delta_{12} . \mathbf{X_2} = 0 \\ \delta_{20} + \delta_{21} . \mathbf{X_1} + \delta_{22} . \mathbf{X_2} = 0 \end{cases} \Rightarrow \begin{cases} -582{,}356 + 4{,}43295 \times \mathbf{X_1} + 3{,}86590 \times \mathbf{X_2} = 0 \\ -857{,}184 + 3{,}86590 \times \mathbf{X_1} + 5{,}52107 \times \mathbf{X_2} = 0 \end{cases}$$

$$\begin{cases} \mathbf{X_1} = \mathbf{-10{,}342} \\ \mathbf{X_2} = \mathbf{162{,}498} \end{cases}$$

- **Reações nos apoios**: $\mathbf{R} = \mathbf{R}^{S0} + \mathbf{X_1} \times \mathbf{R}^{S1} + \mathbf{X_2} \times \mathbf{R}^{S2}$

$$\mathbf{V_A} = -17{,}5 - 10{,}342 \times \frac{1}{6} + 162{,}498 \times \frac{1}{6} = 7{,}859 \text{ kN} \uparrow$$

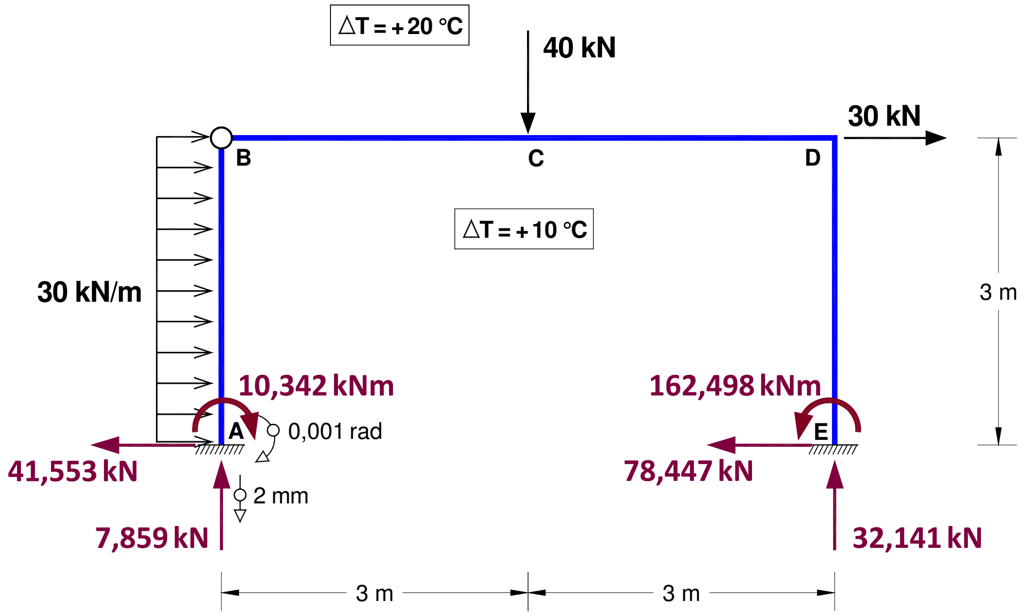
$$\mathbf{H_A} = -45 - 10{,}342 \times \left(-\frac{1}{3}\right) + 162{,}498 \times 0 = -41{,}553 \text{ kN} \leftarrow$$

$$\mathbf{M_A} = 0 - 10{,}342 \times 1 + 162{,}498 \times 0 = -10{,}342 \text{ kNm} \curvearrowright$$

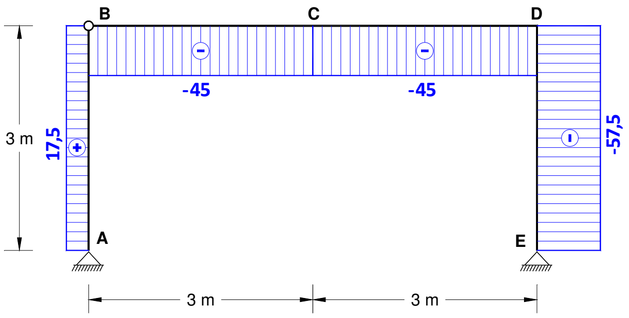
$$\mathbf{V_E} = 57{,}5 - 10{,}342 \times \left(-\frac{1}{6}\right) + 162{,}498 \times \left(-\frac{1}{6}\right) = 32{,}141 \text{ kN} \uparrow$$

$$\mathbf{H_E} = -75 - 10{,}342 \times \frac{1}{3} + 162{,}498 \times 0 = -78{,}447 \text{ kN} \leftarrow$$

$$\mathbf{M_E} = 0 - 10{,}342 \times 0 + 162{,}498 \times 1 = 162{,}498 \text{ kNm} \curvearrowleft$$

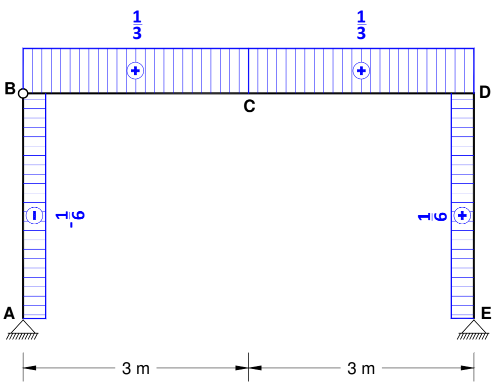


- **Diagrama de esforços axiais nas barras**: $N = N^{S0} + X_1 \times N^{S1} + X_2 \times N^{S2}$



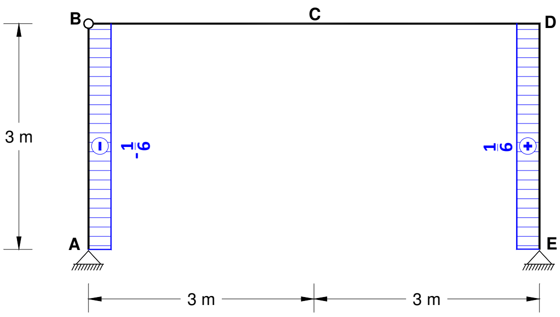
SISTEMA S_0

Esforços Axiais (kN)



SISTEMA S_1

Esforços Axiais (kN)



SISTEMA S_2

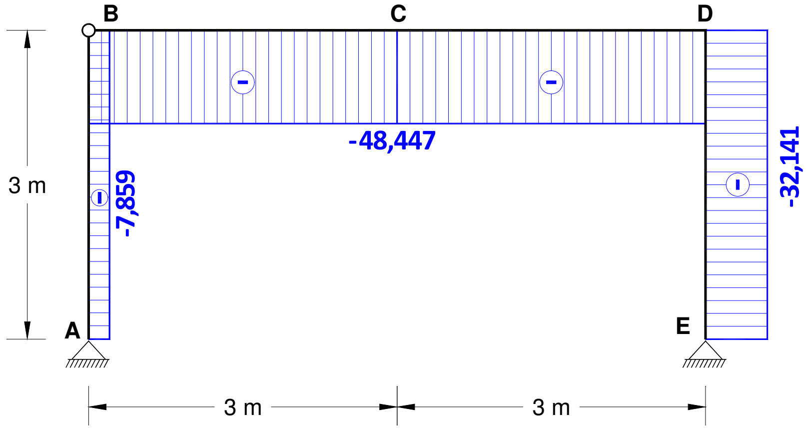
Esforços Axiais (kN)

$$\boxed{N = N^{S0} + X_1 \cdot N^{S1} + X_2 \cdot N^{S2}}$$

Barra AB $\Rightarrow$ $N = -17{,}5 - 10{,}342 \times \left(-\frac{1}{6}\right) + 162{,}498 \times \left(-\frac{1}{6}\right) = -7{,}859$ kN

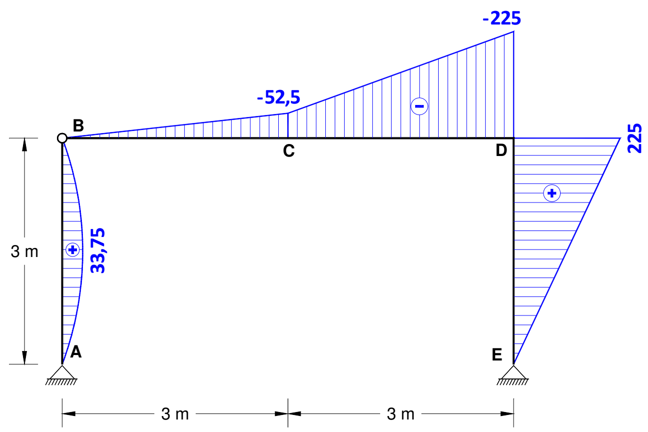
Barra BCD $\Rightarrow$ $N = -45 - 10{,}342 \times \frac{1}{3} + 162{,}498 \times \frac{1}{6} = -48{,}447$ kN

Barra ED $\Rightarrow$ $N = -57{,}5 - 10{,}342 \times \frac{1}{6} + 162{,}498 \times \frac{1}{6} = -32{,}141$ kN



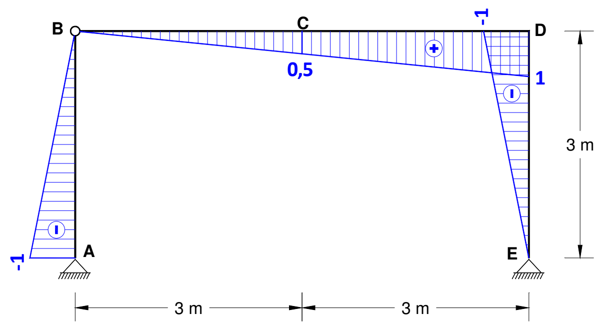
Esforços Axiais (kN)

- **Diagrama de momentos fletores nas barras**: $M = M^{S0} + X_1 \times M^{S1} + X_2 \times M^{S2}$



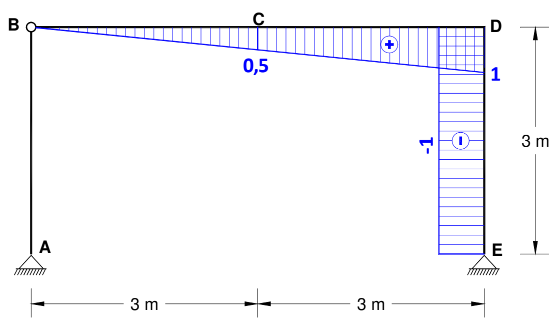
SISTEMA S_0

Momentos Fletores (kNm)



SISTEMA S_1

Momentos Fletores (kNm)



SISTEMA S_2

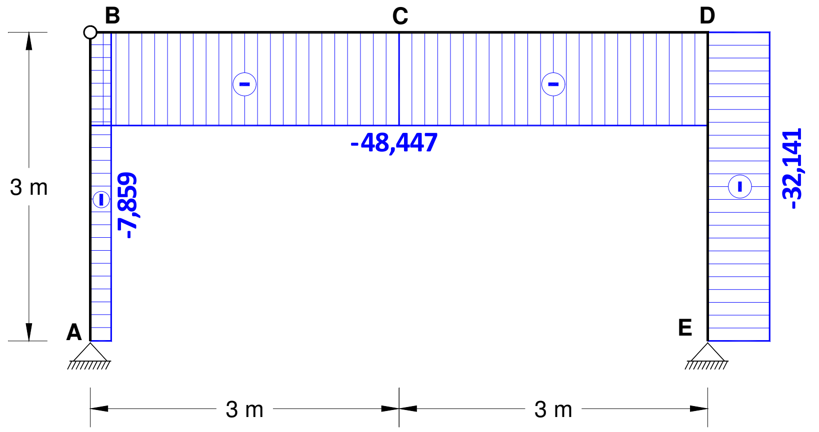
Momentos Fletores (kNm)

$$\boxed{N = N^{S0} + X_1 \cdot N^{S1} + X_2 \cdot N^{S2}}$$

Barra AB $\Rightarrow$ $N = -17{,}5 - 10{,}342 \times \left(-\frac{1}{6}\right) + 162{,}498 \times \left(-\frac{1}{6}\right) = -7{,}859$ kN

Barra BCD $\Rightarrow$ $N = -45 - 10{,}342 \times \frac{1}{3} + 162{,}498 \times \frac{1}{6} = -48{,}447$ kN

Barra ED $\Rightarrow$ $N = -57{,}5 - 10{,}342 \times \frac{1}{6} + 162{,}498 \times \frac{1}{6} = -32{,}141$ kN

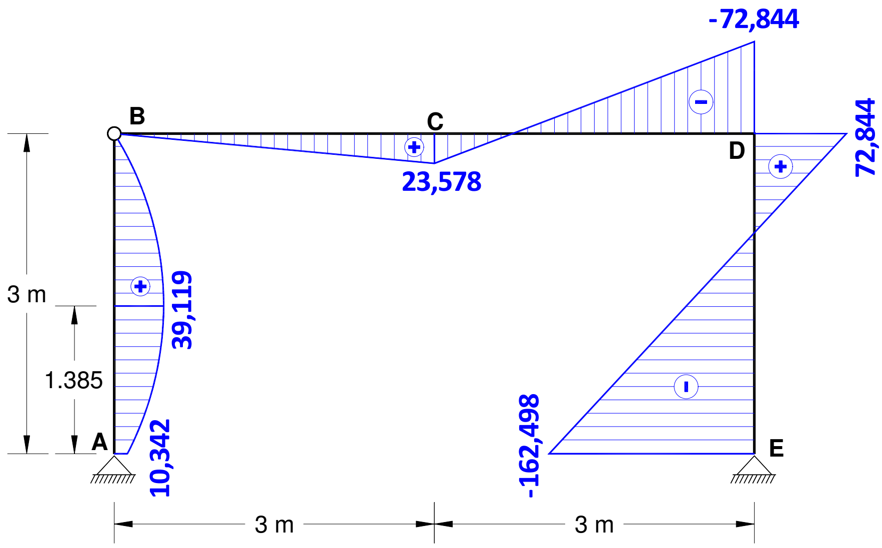


Esforços Axiais (kN)

Barra AB	**Sistema S_0** $\Rightarrow$ $M(z) = 45z - 15z^2$ **Sistema S_1** $\Rightarrow$ $M(z) = -1 + \frac{1}{3}z$ **Sistema S_2** $\Rightarrow$ $M(z) = 0$ $\boxed{M = M^{S0} + X_1 \times M^{S1} + X_2 \times M^{S2}}$ $M(z) = 45z - 15z^2 - 10{,}342 \times (-1 + \frac{1}{3}z) = -15z^2 + \frac{124{,}658}{3}z + 10{,}342$ $\begin{cases} M_{A\,(z=0)} = 10{,}342 \text{ kNm} \\ M_{B\,(z=3)} = 0 \end{cases}$ $M_{máx} \Rightarrow V(z) = 41{,}553 - 30z = 0 \Rightarrow z = 1{,}3851 \Rightarrow M_{máx} = M_{z=1{,}3851} = 39{,}119$ kNm

Barra BCD	$\boxed{M = M^{S0} + X_1 \times M^{S1} + X_2 \times M^{S2}}$ $M_B = 0 - 10{,}342 \times 0 - 162{,}498 \times 0 = 0$ $M_C = -52{,}5 - 10{,}342 \times 0{,}5 - 162{,}498 \times 0{,}5 = 23{,}578$ kNm $M_D = -225 - 10{,}342 \times 1 - 162{,}498 \times 1 = -72{,}844$ kNm

Barra ED	$\boxed{M = M^{S0} + X_1 \times M^{S1} + X_2 \times M^{S2}}$ $M_E = 0 - 10{,}342 \times 0 - 162{,}498 \times (-1) = -162{,}498$ kNm $M_D = 225 - 10{,}342 \times (-1) - 162{,}498 \times (-1) = 72{,}844$ kNm



Momentos Fletores (kNm)

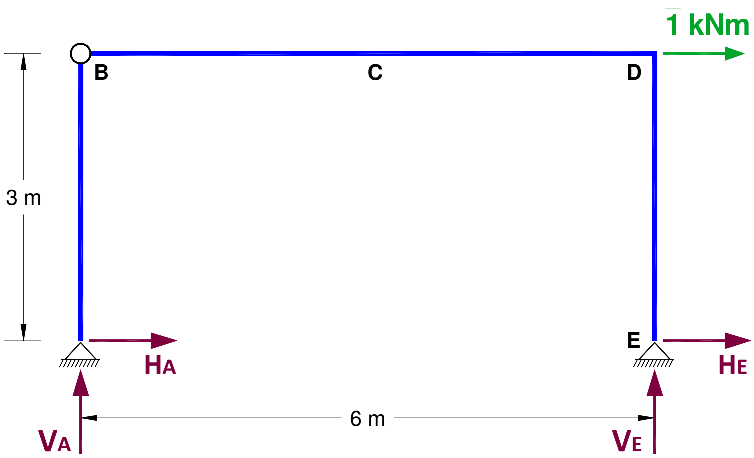
ALÍNEA b - Deslocamento horizontal da secção D

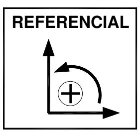
• Sistema virtual $\overline{S}$

Equações de equilíbrio
(arco de 3 rótulas)

Toda a estrutura $\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \end{cases}$

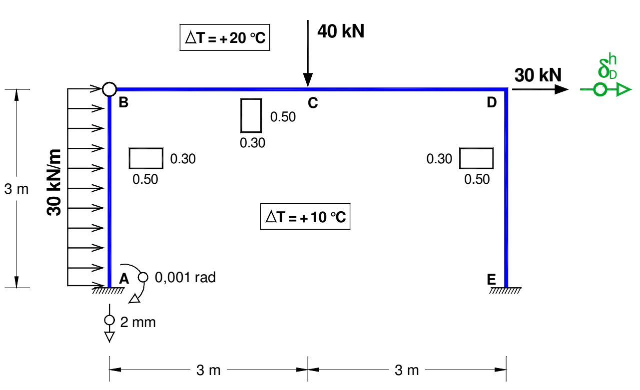
Corpo $\overline{AB}$ $\quad \sum M^{AB.}_{rot.B} = 0$



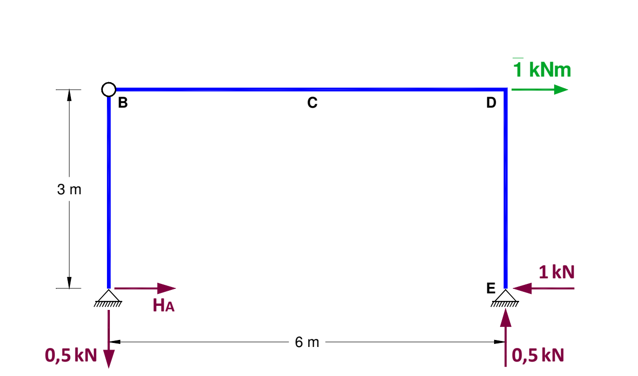


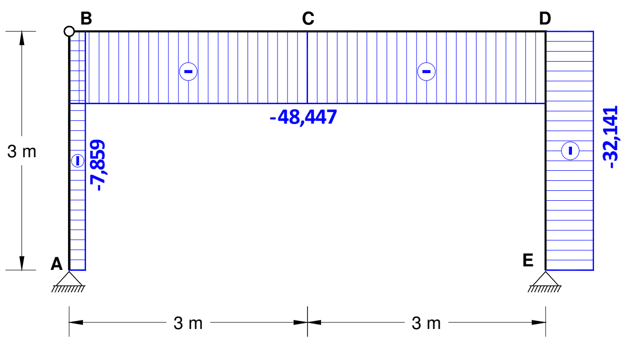
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_E = 0 \\ \sum M^{AB.}_{rot.B} = 0 \end{cases} \Rightarrow \begin{cases} H_A + H_E + 1 = 0 \\ V_A + V_E = 0 \\ -V_A \times 6 - 1 \times 3 = 0 \\ H_A \times 3 = 0 \end{cases} \Rightarrow \begin{cases} H_A = 0 \\ V_A = -0{,}5 \text{ kN} \downarrow \\ H_E = -1 \text{ kN} \leftarrow \\ V_E = 0{,}5 \text{ kN} \uparrow \end{cases}$$

Sistema real S

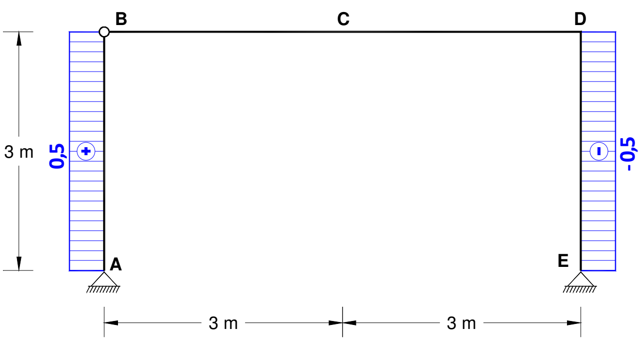


Sistema virtual $\bar{S}$

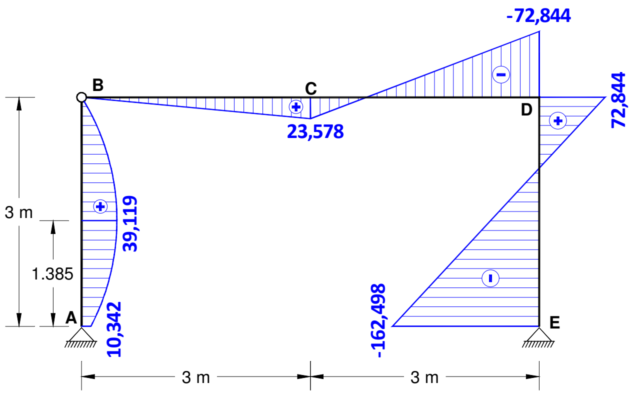




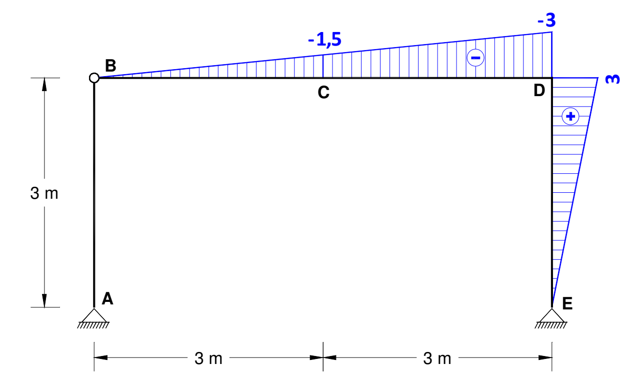
Esforços axiais (kN)



Esforços Axiais (kN)



Momentos fletores (kNm)



Momentos fletores (kNm)

Vamos aplicar o TTV com **sistema real S** e **o sistema virtual $\bar{S}$** $\Rightarrow$ $\mathbf{W_{ext} = W_{int}}$

$$\boxed{\sum F_{ext\,\bar{S}} \times \Delta_R = \int \frac{\bar{N}.N}{E\,A}\,dz + \int \frac{\bar{M}.M}{E\,I}\,dz + \int \bar{N}\,.\,\alpha\,.\,T_{méd}\,dz + \int \bar{M}\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}\,dz}$$

$$\sum F_{ext\,\bar{S}} \times \Delta_R = \bar{1} \times \delta_D^h + 0{,}5 \times 0{,}002$$

Barra AB $\Rightarrow \int \frac{\bar{N}.N}{EA} dz = \frac{1}{EA} \times 0{,}5 \times (-7{,}859) = \frac{1}{4{,}35 \times 10^6} \times (-11{,}7885)$ Barras BCD $\Rightarrow \int \frac{\bar{N}.N}{EA} dz = 0$ Barra ED $\Rightarrow \int \frac{\bar{N}.N}{EA} dz = \frac{1}{EA} \times (-0{,}5) \times (-32{,}141) \times 3 = \frac{1}{4{,}35 \times 10^6} \times 48{,}2115$	$\int \frac{\bar{N}.N}{EA} dz = \frac{36{,}423}{4{,}35 \times 10^6}$

Barra AB $\Rightarrow \int \frac{\bar{M}.M}{EI} dz = 0$ Barra BC $\Rightarrow \int \frac{\bar{M}.M}{EI} dz = \frac{1}{EI} \times \frac{(-1{,}5) \times 23{,}578 \times 3}{3} = -\frac{35{,}367}{EI} = -\frac{35{,}367}{90\,625}$ Barra CD $\Rightarrow \int \frac{\bar{M}.M}{EI} dz =$ $= \frac{1}{EI} \times \frac{3}{6} \left(2 \times (-1{,}5) \times 23{,}578 + (-1{,}5) \times (-72{,}844) + (-3) \times 23{,}578 + 2 \times (-1{,}5) \times (-72{,}844)\right) =$ $= \frac{202{,}431}{EI} = \frac{202{,}431}{90\,625}$ Barra ED $\Rightarrow \int \frac{\bar{M}.M}{EI} dz = \frac{1}{EI} \times \frac{3 \times 3}{6} (-162{,}498 + 2 \times 72{,}844) = -\frac{25{,}215}{EI} = -\frac{25{,}215}{90\,625}$	$\int \frac{\bar{M}.M}{EI} dz = \frac{141{,}849}{90\,625}$

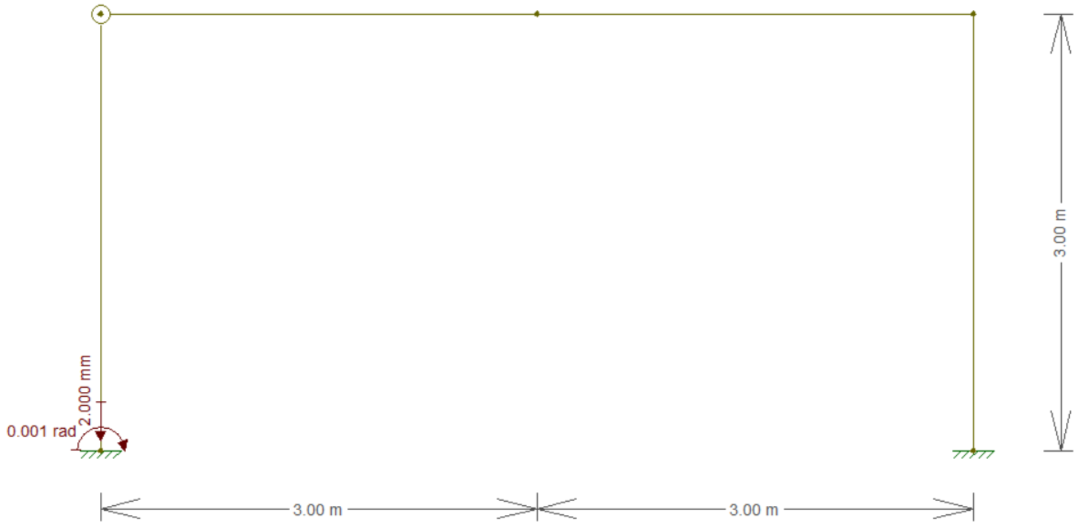
Barra AB $\Rightarrow \int \bar{N} . \alpha . T_{méd}\, dz = 0{,}5 \times 3 \mathrm{x} 10^{-5} \times \frac{20+10}{2} = 22{,}5 \mathrm{x} 10^{-5}$ Barras BCD $\Rightarrow \int \bar{N} . \alpha . T_{méd}\, dz = 0$ Barra DE $\Rightarrow \int \bar{N} . \alpha . T_{méd}\, dz = (-0{,}5) \times 3 \mathrm{x} 10^{-5} \times \frac{20+10}{2} = -22{,}5 \mathrm{x} 10^{-5}$	$\int \bar{N} . \alpha . T_{méd}\, dz = 0$

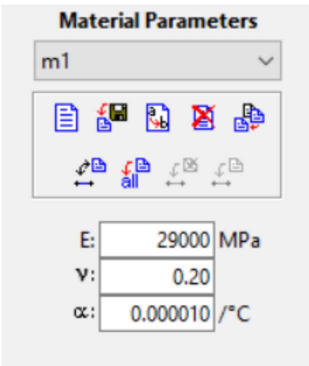
Barra AB $\Rightarrow \int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} dz = 0$ Barras BCD $\Rightarrow \int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} dz = \frac{-3 \times 6}{2} \times 10^{-5} \times \frac{10-20}{0{,}50} = 180 \mathrm{x} 10^{-5}$ Barra DE $\Rightarrow \int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} dz = 4{,}5 \times 10^{-5} \times \frac{20-10}{0{,}50} = 90 \mathrm{x} 10^{-5}$	$\int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} dz = 270 \mathrm{x} 10^{-5}$

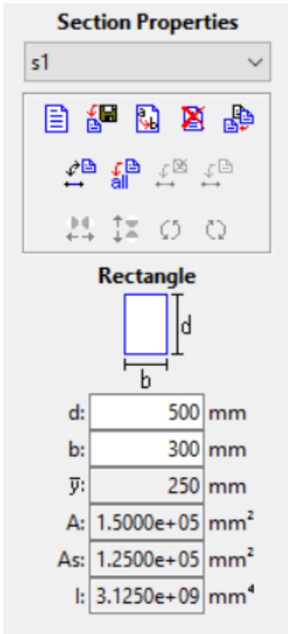
$$\boxed{\sum F_{ext\,\bar{S}} \times \Delta_R = \int \frac{\bar{N}.N}{EA} dz + \int \frac{\bar{M}.M}{EI} dz + \int \bar{N} . \alpha . T_{méd}\, dz + \int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} dz}$$

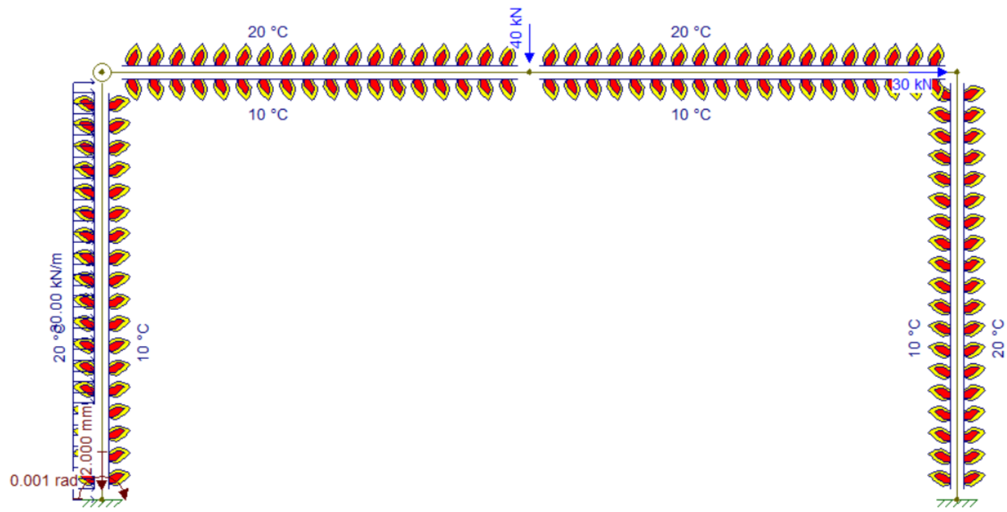
$$\bar{1} \times \delta_D^h + 0{,}5 \times 0{,}002 = \frac{36{,}423}{4{,}35 \times 10^6} + \frac{141{,}849}{90\,625} + 0 + 270 \mathrm{x} 10^{-5} \Rightarrow \delta_D^h = 3{,}274 \mathrm{x} 10^{-3}\ \mathrm{m} = 3{,}274\ \mathrm{mm} \rightarrow$$

ALÍNEA c - FTOOL



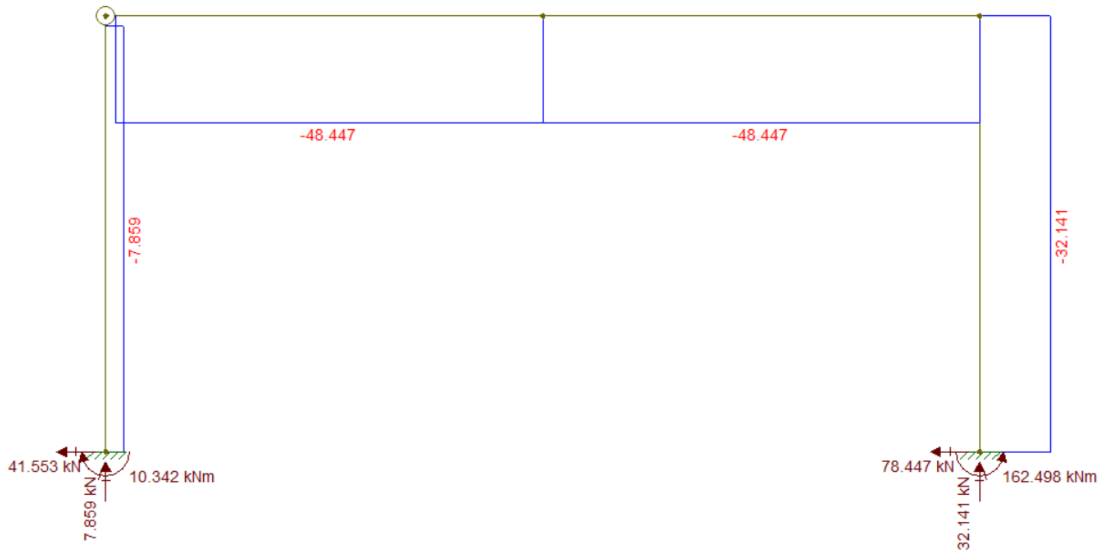




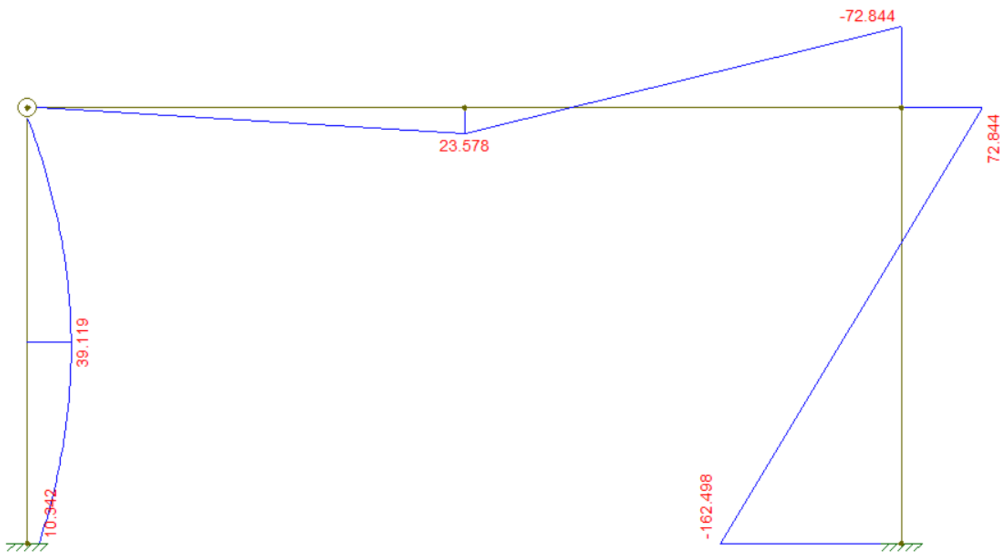


ESFORÇOS AXIAIS
(kN)

REAÇÕES



MOMENTOS FLETORES
(kNm)



DEFORMADA