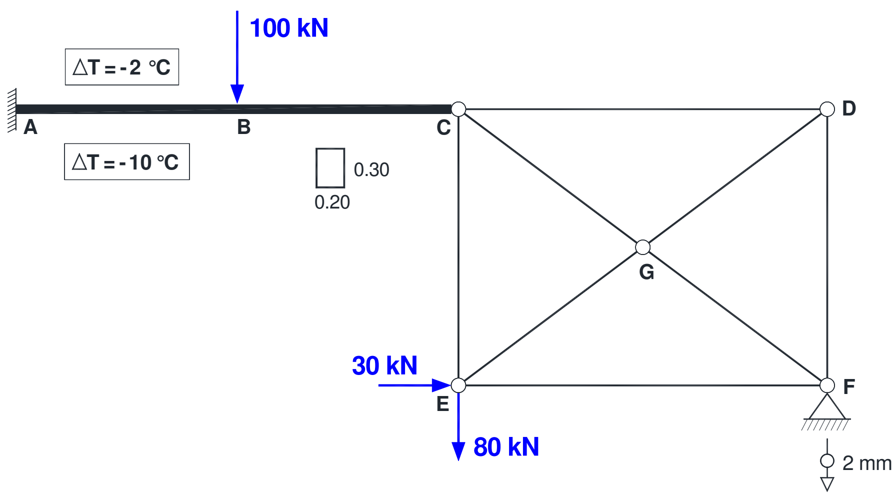
LICENCIATURA EM ENGENHARIA CIVIL

TEORIA DE ESTRUTURAS

MÉTODO DAS FORÇAS



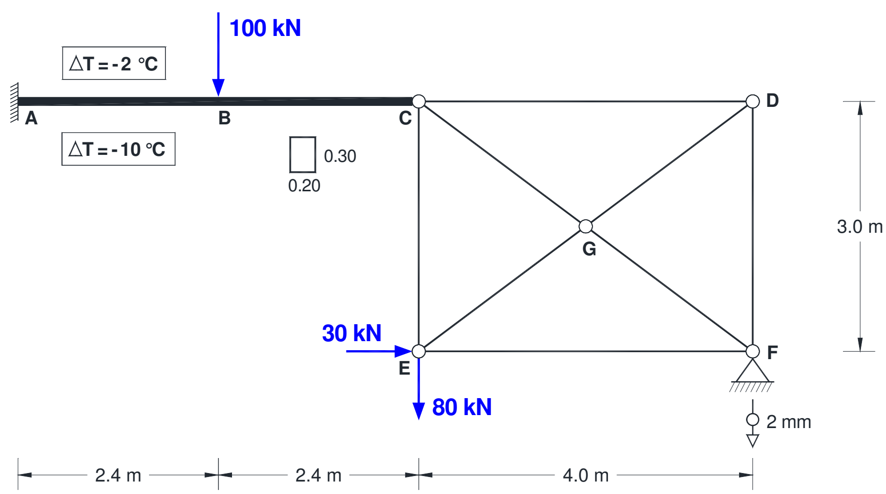
ESTRUTURA MISTA HIPERESTÁTICA

ISABEL ALVIM TELES

EXERCÍCIO

Considere a estrutura abaixo representada.

As barras **AB** e **BC** estão submetidas às variações de temperatura indicadas na figura e o apoio **F** tem um assentamento vertical de 2 mm.



Barras AB e **BC** Secção (bxh) = 0,20m x 0,30m Betão: E = 30 GPa; $\alpha = 10^{-5}$ /°C
Restantes barras Área = 10 cm^2 Aço: E = 210 GPa

Responda às alíneas seguintes desprezando a contribuição do esforço transverso.

a) Determine as reações e trace os diagramas de esforços da estrutura;

b) Determine o deslocamento vertical do ponto **B**.

RESOLUÇÃO

Alínea a)

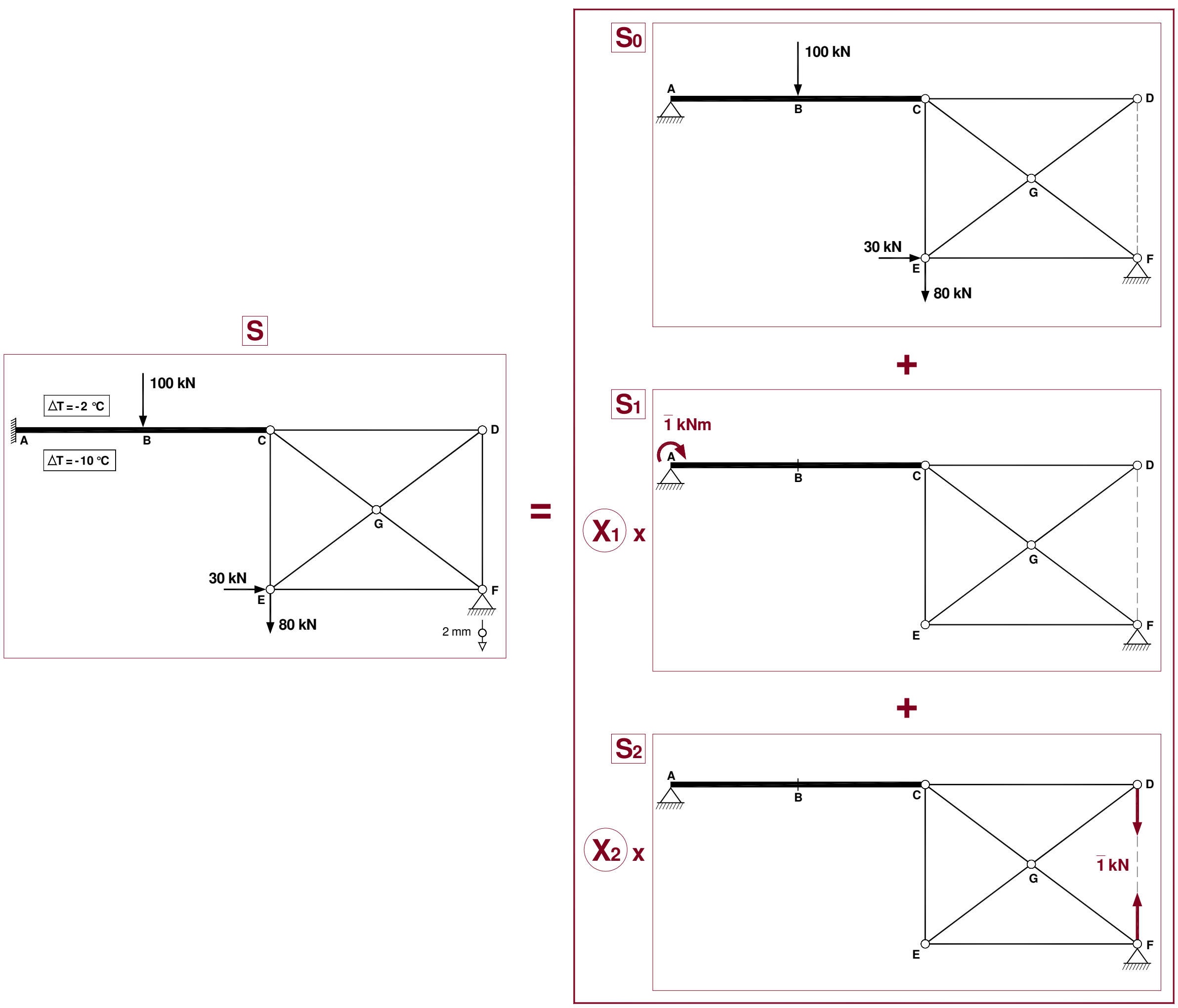
A estrutura é 1 vez hiperestática por condições externas e 1 vez hiperestática por condições internas, logo a estrutura é **hiperestática de grau 2**.

A estrutura (**S**) vai ser decomposta no sistema $\mathbf{S_0}$ e nos sistemas $\mathbf{S_1}$ e $\mathbf{S_2}$:

$$\mathbf{S} = \mathbf{S_0} + \mathbf{X_1} \times \mathbf{S_1} + \mathbf{X_2} \times \mathbf{S_2}$$

sendo: $\mathbf{X_1}$ – incógnita hiperestática correspondente ao momento de encastramento no apoio **A**.

$\mathbf{X_2}$ – incógnita hiperestática correspondente ao esforço axial na barra **DF**.

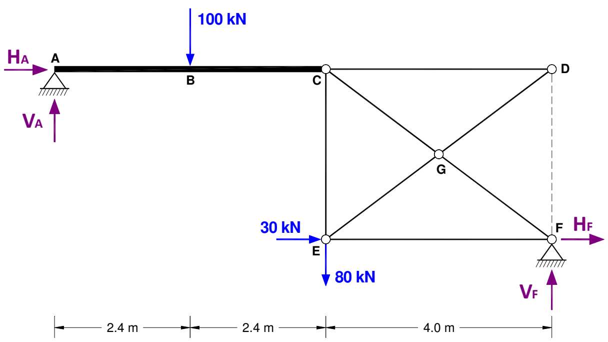


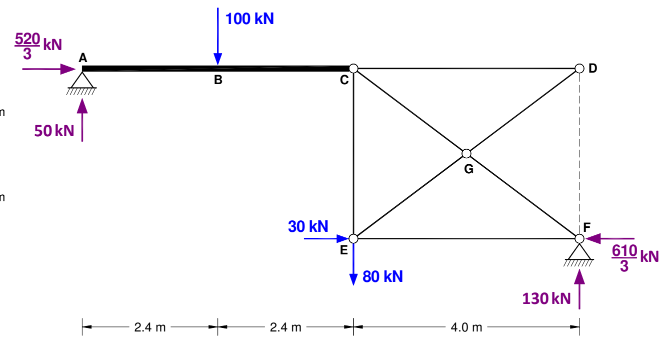
• Cálculo da estrutura S_0

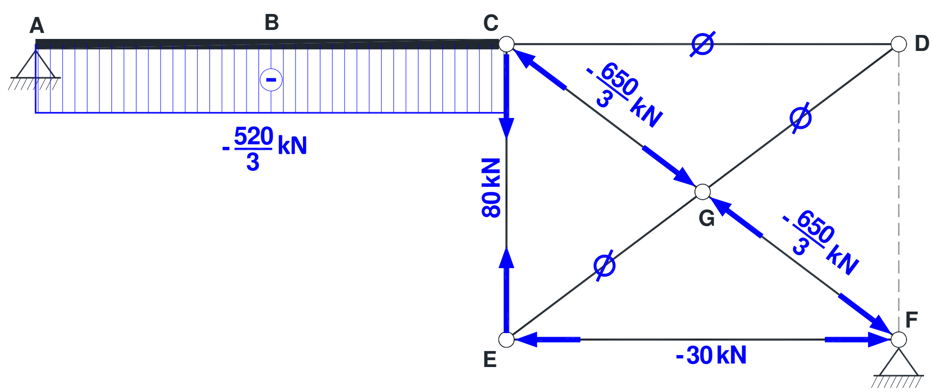
Determinação das reações:

Toda a estrutura	$\sum Fx=0$
	$\sum Fy=0$
	$\sum M_A=0$
Corpo **ABC**	$\sum M_C=0$

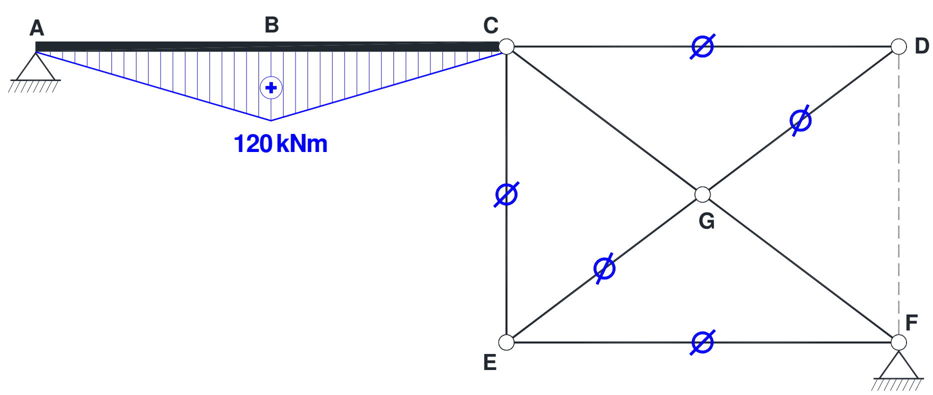
$$\begin{cases} H_A + H_F + 30 = 0 \\ V_A + V_F - 180 = 0 \\ V_F \times 8{,}8 + H_F \times 3 + 30 \times 3 - 80 \times 4{,}8 - 100 \times 2{,}4 = 0 \\ 100 \times 2{,}4 - V_A \times 4{,}8 = 0 \end{cases} \qquad \begin{cases} H_A = \dfrac{520}{3}\ \text{kN} \ \rightarrow \\ V_A = 50\ \text{kN} \ \uparrow \\ H_F = -\dfrac{610}{3}\ \text{kN} \ \leftarrow \\ V_F = 130\ \text{kN} \ \uparrow \end{cases}$$







Sistema base S_0 - Esforços axiais



Sistema base S_0 - Momentos fletores

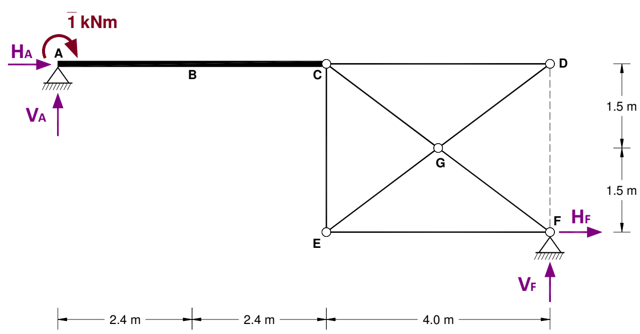
• Cálculo da estrutura S_1

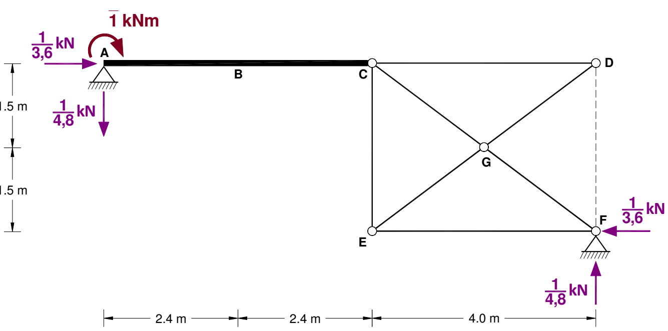
Determinação das reações:

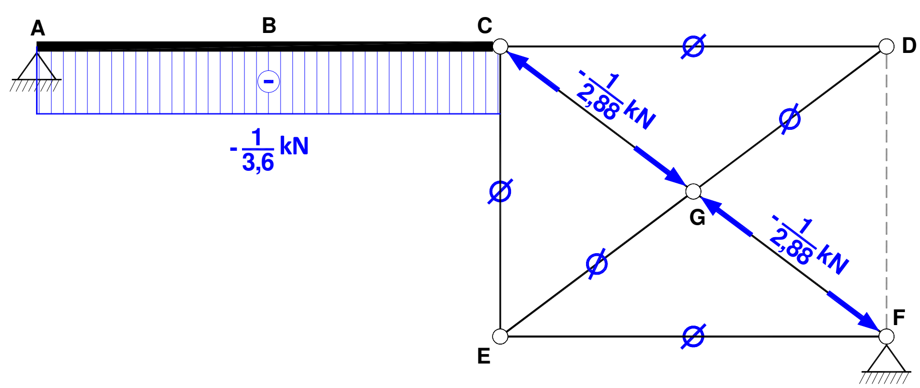
Toda a estrutura	$\sum Fx=0$
	$\sum Fy=0$
	$\sum M_A=0$
Corpo **ABC**	$\sum M_C=0$

$$\begin{cases} H_A + H_F = 0 \\ V_A + V_F = 0 \\ V_F \times 8{,}8 + H_F \times 3 - 1 = 0 \\ 1 + V_A \times 4{,}8 = 0 \end{cases}$$

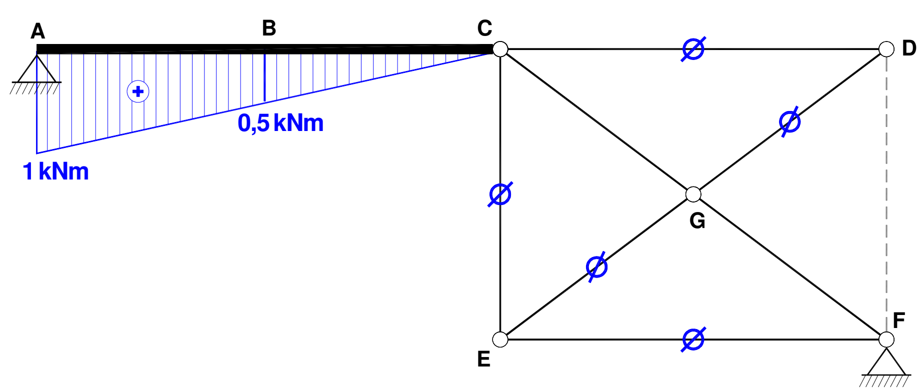
$$\begin{cases} H_A = \dfrac{1}{3{,}6}\ \text{kN} \rightarrow \\ V_A = -\dfrac{1}{4{,}8}\ \text{kN} \downarrow \\ H_F = -\dfrac{1}{3{,}6}\ \text{kN} \leftarrow \\ V_F = \dfrac{1}{4{,}8}\ \text{kN} \uparrow \end{cases}$$







Sistema base S_1 - Esforços axiais

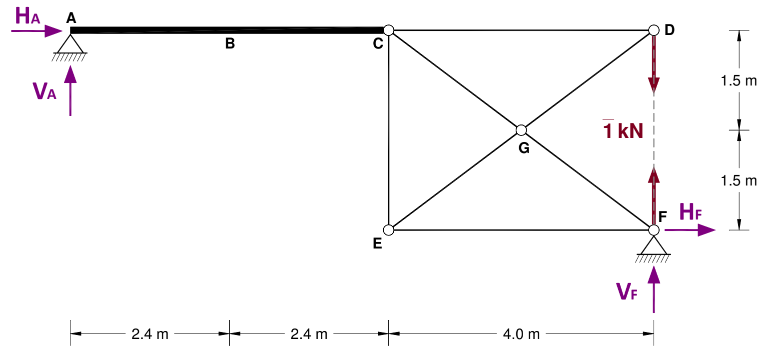


Sistema base S_1 - Momentos fletores

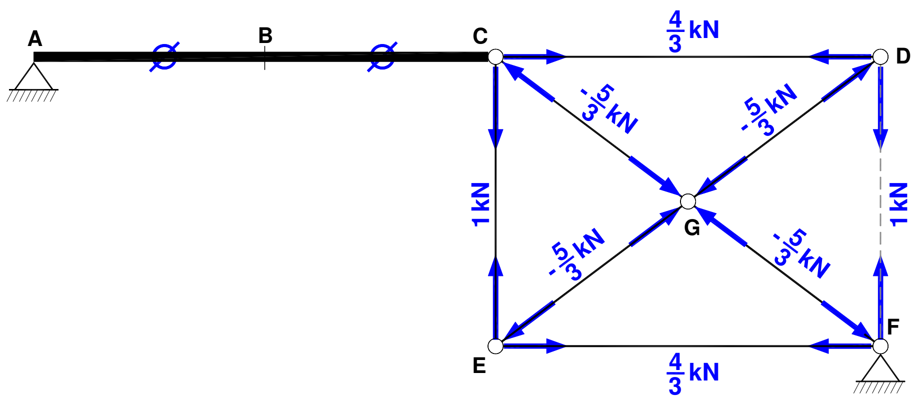
• Cálculo da estrutura S_2

Determinação das reações:

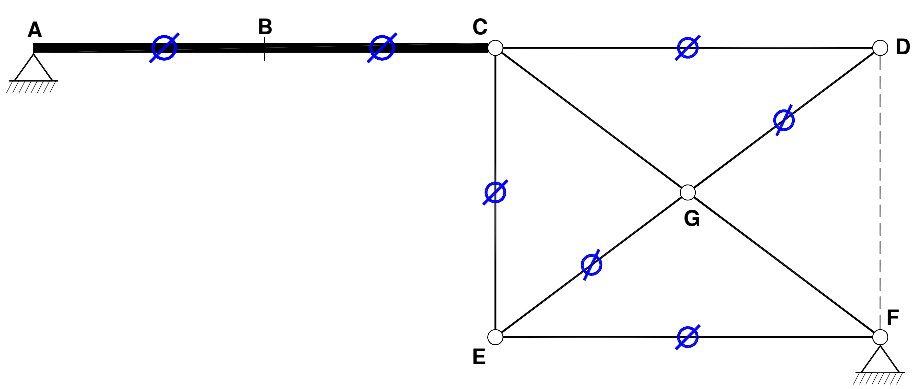
Toda a estrutura	$\sum Fx=0$ $\sum Fy=0$ $\sum M_A=0$
Corpo **ABC**	$\sum M_C=0$



$$\begin{cases} H_A + H_F = 0 \\ V_A + V_F = 0 \\ V_F \times 8{,}8 + H_F \times 3 = 0 \\ V_A \times 4{,}8 = 0 \end{cases} \qquad \begin{cases} H_A = 0 \\ V_A = 0 \\ H_F = 0 \\ V_F = 0 \end{cases}$$



Sistema base S_2 - Esforços axiais



Sistema base S_2 - Momentos fletores

MÉTODO DAS FORÇAS

(desprezando a contribuição do Esforço Transverso)

Barras bi-articuladas | **Barras contínuas**

$$\begin{cases} \Sigma F_{ext}^{S1} \times \Delta_R = \Sigma N_1 \dfrac{N L}{E A} + \Sigma N_1\, \alpha\, \Delta T\, L + \int \dfrac{N_1 \times N}{EA} dz + \int \dfrac{M_1 \times M}{E I} dz + \int N_1 . \alpha . T_{med}\, dz + \int M_1 . \alpha \dfrac{T_{inf} - T_{sup}}{h} dz \\ \Sigma F_{ext}^{S2} \times \Delta_R = \Sigma N_2 \dfrac{N L}{E A} + \Sigma N_2\, \alpha\, \Delta T\, L + \int \dfrac{N_2 \times N}{EA} dz + \int \dfrac{M_2 \times M}{E I} dz + \int N_2 . \alpha . T_{med}\, dz + \int M_2 . \alpha \dfrac{T_{inf} - T_{sup}}{h} dz \end{cases}$$

Considerando: $\begin{cases} N = N_0 + X_1 N_1 + X_2 N_2 \\ M = M_0 + X_1 M_1 + X_2 M_2 \end{cases}$

$$\begin{cases} \delta_{10} + \delta_{11} . \mathbf{X_1} + \delta_{12} . \mathbf{X_2} = 0 \\ \delta_{20} + \delta_{21} . \mathbf{X_1} + \delta_{22} . \mathbf{X_2} = 0 \end{cases}$$

barras bi-articuladas

$$\delta_{10} = -\Sigma F_{ext}^{S1} \times \Delta_R + \Sigma N_1 \frac{N_0 L}{E A} + \Sigma N_1\, \alpha\, \Delta T\, L +$$
$$+ \int \frac{N_1 \times N_0}{EA} dz + \int N_1 . \alpha . T_{med}\, dz + \int \frac{M_1 \times M_0}{E I} dz + \int M_1 . \alpha \frac{T_{inf} - T_{sup}}{h} dz$$

barras contínuas

barras bi-articuladas

$$\delta_{20} = -\Sigma F_{ext}^{S2} \times \Delta_R + \Sigma N_2 \frac{N_0 L}{E A} + \Sigma N_2\, \alpha\, \Delta T\, L +$$
$$+ \int \frac{N_2 \times N_0}{EA} dz + \int N_2 . \alpha . T_{med}\, dz + \int \frac{M_2 \times M_0}{E I} dz + \int M_2 . \alpha \frac{T_{inf} - T_{sup}}{h} dz$$

barras contínuas

artic. | cont.

$$\delta_{11} = \Sigma N_1 \frac{N_1 L}{E A} + \int \frac{N_1 \times N_1}{EA} dz + \int \frac{M_1 \times M_1}{E I} dz$$

artic. | cont.

$$\delta_{12} = \Sigma N_1 \frac{N_2 L}{E A} + \int \frac{N_1 \times N_2}{EA} dz + \int \frac{M_1 \times M_2}{E I} dz = \delta_{21}$$

artic. | cont.

$$\delta_{22} = \Sigma N_2 \frac{N_2 L}{E A} + \int \frac{N_2 \times N_2}{EA} dz + \int \frac{M_2 \times M_2}{E I} dz$$

• Barras bi-articuladas

$$\mathbf{E\,A} = 210 \times 10^{6} \;\; \text{x} \;\; 10 \times 10^{-4} \;=\; 2{,}1 \times 10^{5} \;\; \text{kPa x m}^{2}$$

BARRAS	L (m)	N_0 (kN)	N_1 (kN)	N_2 (kN)	$N_1 \frac{N_0 L}{EA}$	$N_2 \frac{N_0 L}{EA}$	$N_1 \frac{N_1 L}{EA}$	$N_1 \frac{N_2 L}{EA}$	$N_2 \frac{N_2 L}{EA}$
CD	4	0	0	$\frac{4}{3}$	0	0	0	0	$3{,}386\text{x}10^{-5}$
EF	4	-30	0	$\frac{4}{3}$	0	$-7{,}619\text{x}10^{-4}$	0	0	$3{,}386\text{x}10^{-5}$
CE	3	80	0	1	0	$1{,}143\text{x}10^{-3}$	0	0	$1{,}429\text{x}10^{-5}$
CG	2,5	$-\frac{650}{3}$	$-\frac{1}{2{,}88}$	$-\frac{5}{3}$	$8{,}956\text{x}10^{-4}$	$4{,}299\text{x}10^{-3}$	$1{,}435\text{x}10^{-6}$	$6{,}889\text{x}10^{-6}$	$3{,}307\text{x}10^{-5}$
GF	2,5	$-\frac{650}{3}$	$-\frac{1}{2{,}88}$	$-\frac{5}{3}$	$8{,}956\text{x}10^{-4}$	$4{,}299\text{x}10^{-3}$	$1{,}435\text{x}10^{-6}$	$6{,}889\text{x}10^{-6}$	$3{,}307\text{x}10^{-5}$
EG	2,5	0	0	$-\frac{5}{3}$	0	0	0	0	$3{,}307\text{x}10^{-5}$
GD	2,5	0	0	$-\frac{5}{3}$	0	0	0	0	$3{,}307\text{x}10^{-5}$
DF	3	-	-	1	0	0	0	0	$1{,}429\text{x}10^{-5}$
Σ					$1{,}791\text{x}10^{-3}$	$8{,}979\text{x}10^{-3}$	$2{,}871\text{x}10^{-6}$	$1{,}378\text{x}10^{-5}$	$2{,}286\text{x}10^{-4}$

• Barras contínuas

$$E\,I = 30\times10^{6}\times\frac{0{,}20\times0{,}30^{3}}{12} = 13\,500 \text{ kPa}\times\text{m}^{4}$$

$$EA = 30\times10^{6}\times0{,}20\times0{,}30 = 1{,}8\times10^{6} \text{ kPa}\times\text{m}^{2}$$

$$T_{méd} = \frac{-10-2}{2} = -6\,^{o}\text{C}$$

$$\frac{T_{inf}-T_{sup}}{h} = \frac{-10-(-2)}{0{,}30} = -\frac{8}{0{,}30}$$

$$\int \frac{N_1\times N_0}{EA}\,dz = 2\times\frac{1}{1{,}8\times10^{6}}\times\left(-\frac{1}{3{,}6}\right)\times\left(-\frac{520}{3}\right)\times2{,}4 = 128{,}395\times10^{-6}$$

$$\int N_1\,.\,\alpha\,.\,T_{méd}\,dz = 2\times\left(-\frac{1}{3{,}6}\right)\times2{,}4\times10^{-5}\times(-6) = 8\times10^{-5}$$

$$\int \frac{M_1\times M_0}{E\,I}\,dz = \frac{1}{13\,500}\left[\frac{120\times2{,}40}{6}\times(2\times0{,}5+1)) + \frac{120\times0{,}5}{3}\times2{,}40\right] = \frac{144}{13\,500} = 1{,}0667\times10^{-2}$$

$$\int M_1\,.\,\alpha\,\frac{T_{inf}-T_{sup}}{h}\,dz = -\frac{8}{0{,}30}\times10^{-5}\times\int M_1\,dz = -\frac{8}{0{,}30}\times10^{-5}\times\frac{4{,}80\times1}{2} = -64\times10^{-5}$$

$$\delta_{10} = -\sum F_{ext}^{S1}\times\Delta_R + \sum N_1\frac{N_0 L}{E\,A} + \sum N_1\,\alpha\,\Delta T\,L \;+$$

$$+\int \frac{N_1\times N_0}{EA}\,dz + \int N_1.\alpha.T_{med}\,dz + \int \frac{M_1\times M_0}{E\,I}\,dz + \int M_1\,.\,\alpha\,\frac{T_{inf}-T_{sup}}{h}\,dz =$$

$$= -\frac{1}{4{,}8}\times(-0{,}002) + 1{,}791\times10^{-3} + 0 + 128{,}395\times10^{-6} + 8\times10^{-5} + 1{,}0667\times10^{-2} - 64\times10^{-5} =$$

$$= 1{,}244\times10^{-2}$$

$$\int \frac{N_2 \times N_0}{EA} dz = 0$$

$$\int N_2 \, . \, \alpha \, . \, T_{méd} \, dz = 0$$

$$\int \frac{M_2 \times M_0}{EI} dz = 0$$

$$\int M_2 \, . \, \alpha \frac{T_{inf} - T_{sup}}{h} dz = 0$$

$$\delta_{20} = - \Sigma F_{ext}^{S2} \times \Delta_R + \Sigma N_2 \frac{N_0 L}{EA} + \Sigma N_2 \alpha \Delta T L +$$

$$+ \int \frac{N_2 \times N_0}{EA} dz + \int N_2 . \alpha . T_{med} \, dz + \int \frac{M_2 \times M_0}{EI} dz + \int M_2 \, . \, \alpha \frac{T_{inf} - T_{sup}}{h} dz =$$

$$= 0 + 8{,}979 \times 10^{-3} + 0 + 0 + 0 + 0 + 0 = 8{,}979 \times 10^{-3}$$

$$\int \frac{N_1 \times N_1}{EA} dz = \frac{1}{1{,}8 \times 10^6} \times (-\frac{1}{3{,}6})^2 \times 4{,}8 = 2{,}0576 \times 10^{-7}$$

$$\int \frac{M_1 \times M_1}{EI} dz = \frac{1}{13\,500} (\frac{1 \times 1 \times 4{,}8}{3}) = 1{,}1852 \times 10^{-4}$$

$$\delta_{11} = \Sigma N_1 \frac{N_1 L}{EA} + \int \frac{N_1 . N_1}{EA} dz + \int \frac{M_1 . M_1}{EI} dz = 2{,}871 \times 10^{-6} + 2{,}0576 \times 10^{-7} + 1{,}1852 \times 10^{-4} = 1{,}216 \times 10^{-4}$$

$$\int \frac{N_1 \times N_2}{EA} dz = \int \frac{N_2 \times N_1}{EA} dz = 0$$

$$\int \frac{M_1 \times M_2}{EI} dz = \int \frac{M_2 \times M_1}{EI} dz = 0$$

$$\delta_{12} = \delta_{21} = \Sigma N_1 \frac{N_2 L}{EA} + \int \frac{N_1 \times N_2}{EA} dz + \int \frac{M_1 \times M_2}{EI} dz = 1{,}378 \times 10^{-5} + 0 + 0 = 1{,}378 \times 10^{-5}$$

$$\int \frac{N_2 \times N_2}{EA} dz = 0$$

$$\int \frac{M_2 \times M_2}{EI} dz = 0$$

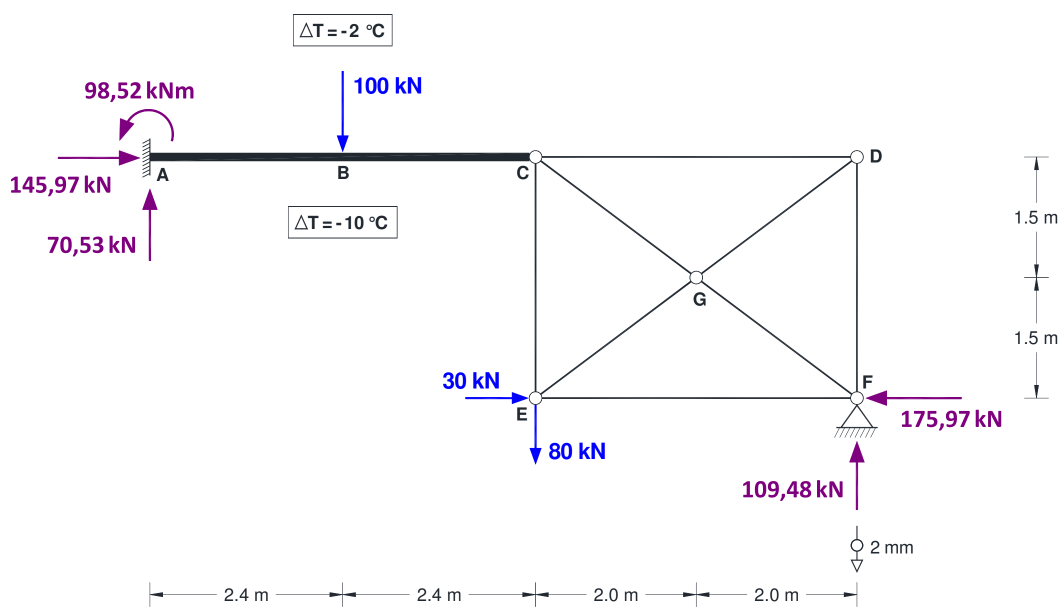
$$\delta_{22} = \Sigma N_2 \frac{N_2 L}{EA} + \int \frac{N_2 \times N_2}{EA} dz + \int \frac{M_2 \times M_2}{EI} dz = 2{,}284 \times 10^{-4} + 0 + 0 = 2{,}284 \times 10^{-4}$$

$$\begin{cases} \delta_{10} + \delta_{11}.\,\mathbf{X_1} + \delta_{12}.\,\mathbf{X_2} = 0 \\ \delta_{20} + \delta_{21}.\,\mathbf{X_1} + \delta_{22}.\,\mathbf{X_2} = 0 \end{cases} \Rightarrow \begin{cases} 1{,}244\times10^{-2} + 1{,}216\times10^{-4}.\,\mathbf{X_1} + 1{,}378\times10^{-5}.\,\mathbf{X_2} = 0 \\ 8{,}979\times10^{-3} + 1{,}378\times10^{-5}.\,\mathbf{X_1} + 2{,}284\times10^{-4}.\,\mathbf{X_2} = 0 \end{cases}$$

$$\begin{cases} \mathbf{X_1} = -98{,}52 \\ \mathbf{X_2} = -33{,}37 \end{cases}$$

$$\begin{cases} H_A = (H_A)^{S0} + \mathbf{X_1}.(H_A)^{S1} + \mathbf{X_2}.(H_A)^{S2} \\ V_A = (V_A)^{S0} + \mathbf{X_1}.(V_A)^{S1} + \mathbf{X_2}.(V_A)^{S2} \\ M_A = (M_A)^{S0} + \mathbf{X_1}.(M_A)^{S1} + \mathbf{X_2}.(M_A)^{S2} \\ H_F = (H_F)^{S0} + \mathbf{X_1}.(H_F)^{S1} + \mathbf{X_2}.(H_F)^{S2} \\ V_F = (V_F)^{S0} + \mathbf{X_1}.(V_F)^{S1} + \mathbf{X_2}.(V_F)^{S2} \end{cases} \Rightarrow \begin{cases} H_A = \frac{520}{3} + (-98{,}52)\times\frac{1}{3{,}6} + (-33{,}37)\times0 \\ V_A = 50 + (-98{,}52)\times(-\frac{1}{4{,}8}) + (-33{,}37)\times0 \\ M_A = 0 + (-98{,}52)\times1 + (-33{,}37)\times0 \\ H_F = -\frac{610}{3} + (-98{,}52)\times(-\frac{1}{3{,}6}) + (-33{,}37)\times0 \\ V_F = 130 + (-98{,}52)\times\frac{1}{4{,}8} + (-33{,}37)\times0 \end{cases}$$

$$\begin{cases} H_A = 145{,}97\text{ kN} \ \rightarrow \\ V_A = 70{,}53\text{ kN} \ \uparrow \\ M_A = -98{,}52\text{ kNm} \ \circlearrowleft \\ H_F = -175{,}97\text{ kN} \ \leftarrow \\ V_F = 109{,}48\text{ kN} \ \uparrow \end{cases}$$



- **Esforços nas barras**

$$N_{barra} = (N_{barra})^{S0} + \mathbf{X_1}.(N_{barra})^{S1} + \mathbf{X_2}.(N_{barra})^{S2}$$

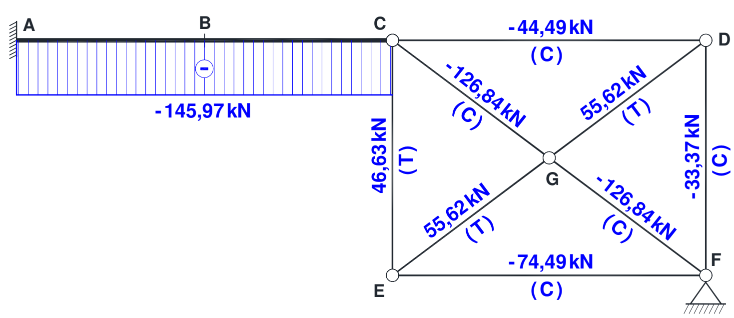
$$N_{barra} = (N_{barra})^{S0} - 98{,}52 \times (N_{barra})^{S1} - 33{,}37 \times (N_{barra})^{S2}$$

Esforços: **-** $\Rightarrow$ compressão
+ $\Rightarrow$ tração

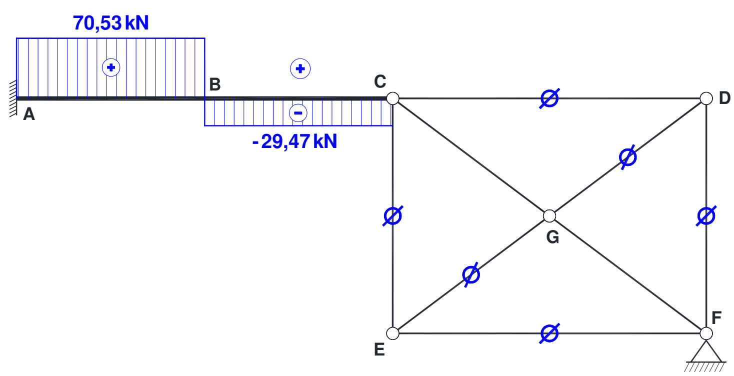
BARRAS	N_0 (kN)	N_1 (kN)	N_2 (kN)	N (kN)
CD	0	0	$\frac{4}{3}$	-44,49
EF	-30	0	$\frac{4}{3}$	-74,49
CE	80	0	1	46,63
CG	$-\frac{650}{3}$	$-\frac{1}{2{,}88}$	$-\frac{5}{3}$	-126,84
GF	$-\frac{650}{3}$	$-\frac{1}{2{,}88}$	$-\frac{5}{3}$	-126,84
EG	0	0	$-\frac{5}{3}$	55,62
GD	0	0	$-\frac{5}{3}$	55,62
DF	0	0	1	-33,37

- **Diagramas de esforços**

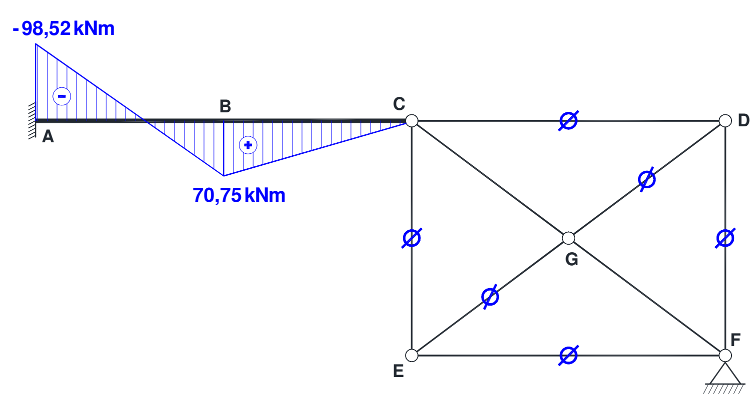
ESFORÇOS AXIAIS



ESFORÇOS TRANSVERSOS



MOMENTOS FLETORES



Alínea b) Determinação do deslocamento vertical do ponto B

TEOREMA DOS TRABALHOS VIRTUAIS (desprezando a contribuição do esforço transverso):

Barras bi-articuladas

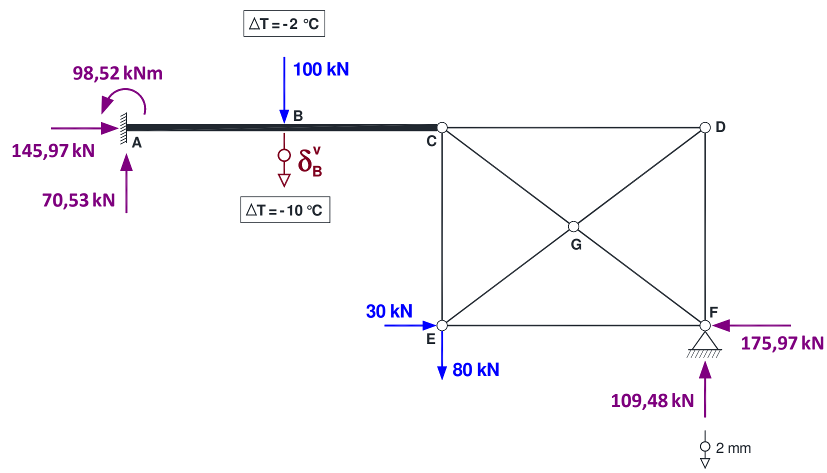
$$\bar{1} \times \delta_B^v + \Sigma\left(\bar{R} \times \text{assent. apoio}\right) = \Sigma \bar{N} \frac{N L}{E A} + \Sigma \bar{N} \alpha \, \Delta T \, L \; +$$

$$+ \int \frac{\bar{N} \times N}{E A} dz + \int \frac{\bar{M} \times M}{E I} dz + \int \bar{N} . \alpha . T_m . dz + \int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} dz$$

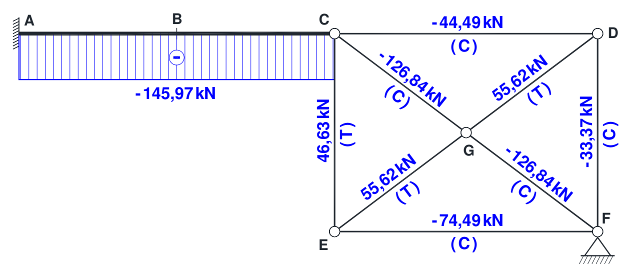
Barras contínuas

SISTEMA REAL

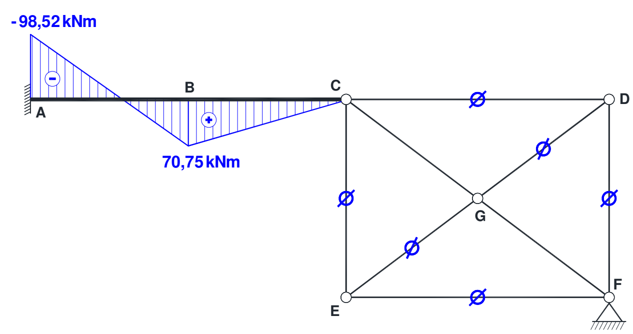
S – SISTEMA REAL



S – Esforço axial (kN)

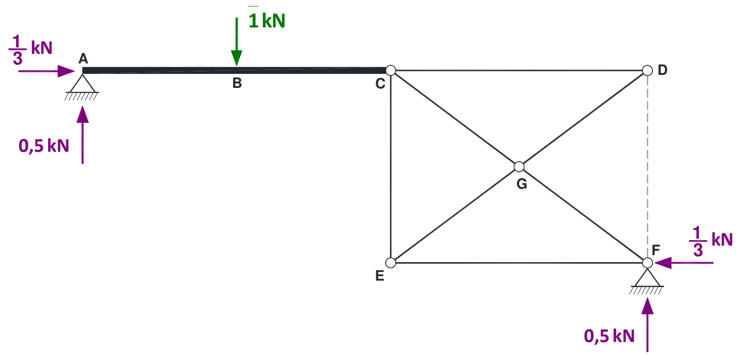


S – Momento fletor (kNm)

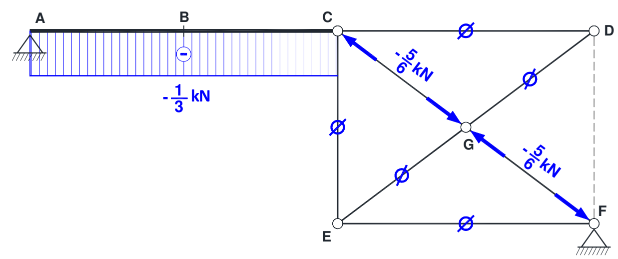


SISTEMA VIRTUAL

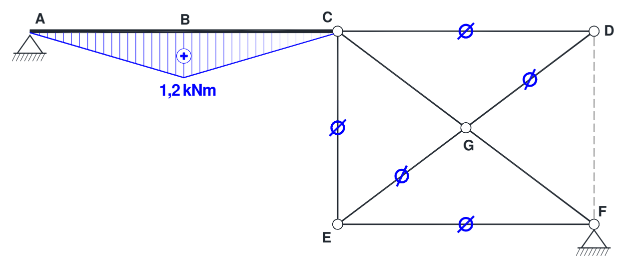
$\bar{S}$ – SISTEMA VIRTUAL



$\bar{S}$ – Esforço axial (kN)



$\bar{S}$ – Momento fletor (kNm)



$$\Sigma\left(\bar{R} \times \text{assent. apoio}\right) = 0{,}5 \times (-0{,}002) = -0{,}001 = -100 \times 10^{-5}$$

$$\Sigma \bar{N} \frac{N\,L}{E\,A} = \frac{1}{2{,}1 \times 10^{5}} (-\frac{5}{6}) \times (-126{,}84) \times 2{,}5 \times 2 = 251{,}667 \times 10^{-5}$$

$$\Sigma \bar{N} \alpha \ \Delta T \ L = 0$$

$$\int \frac{\bar{N} \times N}{E\,A} dz = \frac{1}{1{,}8 \times 10^{6}} \times (-\frac{1}{3}) \times (-145{,}97) \times 4{,}8 = 12{,}975 \times 10^{-5}$$

$$\int \frac{\bar{M} \times M}{E\,I} dz = \frac{1}{13\,500}\left[\frac{1{,}2 \times 2{,}4}{6}(2 \times 70{,}75 - 98{,}52) + \frac{1{,}2 \times 70{,}75 \times 2{,}4}{3}\right] = 655{,}929 \times 10^{-5}$$

$$\int \overline{N}.\alpha.T_{méd}\,dz = (-\frac{1}{3})\times 4{,}8\times 10^{-5}\times(-6) = 9{,}6\times 10^{-5}$$

$$\int \overline{M}.\alpha\frac{T_{inf}-T_{sup}}{h}dz = -\frac{8}{0{,}30}\times 10^{-5}\times\int \overline{M}dz = -\frac{8}{0{,}30}\times 10^{-5}\times\frac{4{,}80\times 1{,}2}{2} = -76{,}8\times 10^{-5}$$

$$\overline{1}\,x\,\delta_B^v + \Sigma\left(\overline{R}\times \text{assent. apoio}\right) = \Sigma\overline{N}\frac{NL}{EA} + \int\frac{\overline{N}\times N}{EA}dz + \int\frac{\overline{M}\times M}{EI}dz + \int \overline{N}.\alpha.T_m.dz + \int \overline{M}.\alpha\frac{T_{inf}-T_{sup}}{h}dz$$

$$\delta_B^v - 100\times 10^{-5} = 251{,}667\times 10^{-5} + 12{,}975\times 10^{-5} + 655{,}929\times 10^{-5} + 9{,}6\times 10^{-5} - 76{,}8\times 10^{-5}$$

$$\delta_B^v = 953{,}371\times 10^{-5}\text{ m} = 9{,}53\text{ mm} \;\updownarrow$$