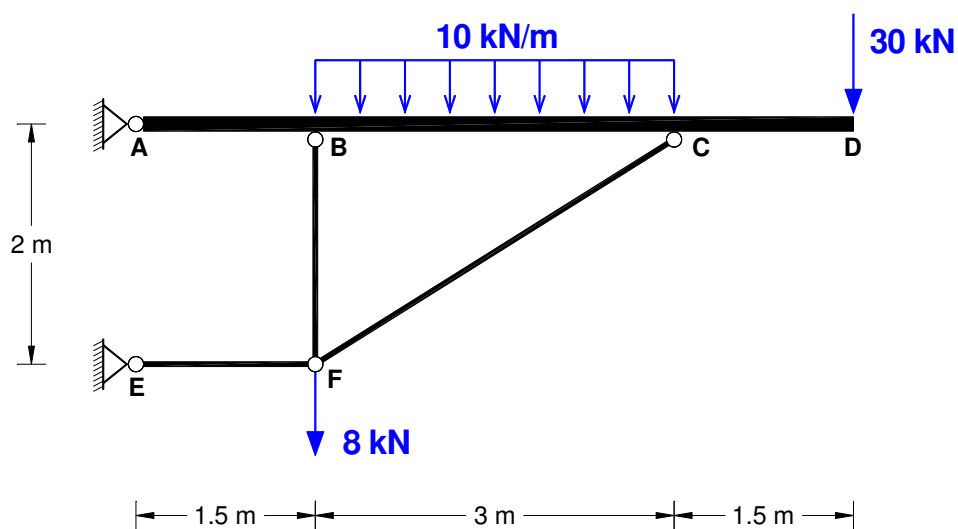


LICENCIATURA EM ENGENHARIA CIVIL

TEORIA DE ESTRUTURAS

TEOREMA DOS TRABALHOS VIRTUAIS

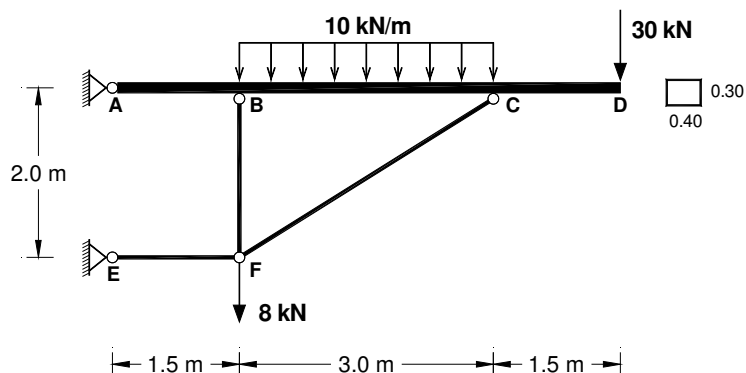


ESTRUTURA MISTA ISOSTÁTICA

ISABEL ALVIM TELES

EXERCÍCIO PROPOSTO

Considere a estrutura representada na figura.



Barras ABCD

Secção: ver figura

Betão: $E = 29 \text{ GPa}$

Restantes barras

Perfil tubular:

100 mm x 100 mm

esp = 5 mm

Aço: $E = 206 \text{ GPa}$

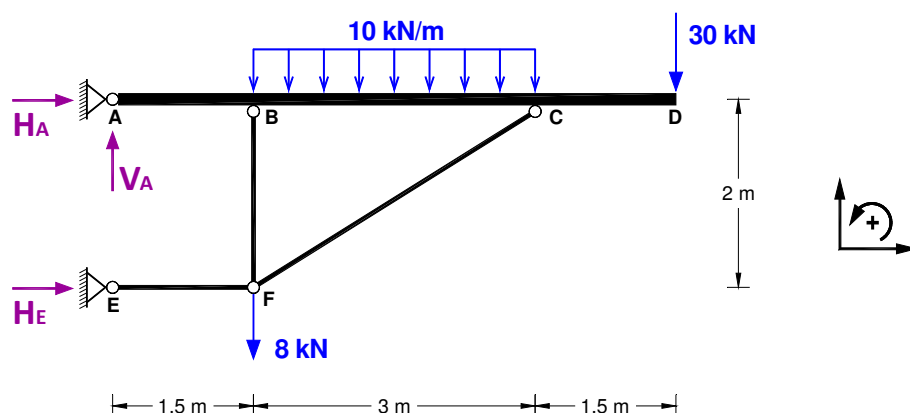


Responda às alíneas seguintes desprezando a contribuição do esforço transversal.

- Determine o deslocamento e rotação do ponto D;
- Determine a rotação do ponto A;
- Determine o deslocamento relativo dos pontos E e B;
- Determine a variação do ângulo formado pelas barras BF e FC;
- Confirme com o programa informático FTOOL os resultados obtidos nas alíneas a) e b).

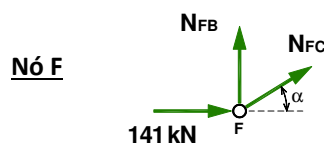
RESOLUÇÃO

• Cálculo das reacções e esforços nas barras bi-articuladas



$$\begin{cases} \sum F_x = 0 & \Rightarrow H_A + H_E = 0 \\ \sum F_y = 0 & \Rightarrow V_A - 30 - 10 \times 3 - 8 = 0 \\ \sum M_A = 0 & \Rightarrow H_E \times 2 - 8 \times 1,5 - 30 \times 6 - 10 \times 3 \times 3 = 0 \end{cases} \Rightarrow \begin{cases} H_A = -141 \text{ kN} \leftarrow \\ V_A = 68 \text{ kN} \uparrow \\ H_E = 141 \text{ kN} \rightarrow \end{cases}$$

Nó E Barra EF $\Rightarrow N_{EF} = -141 \text{ kN}$ (compressão)

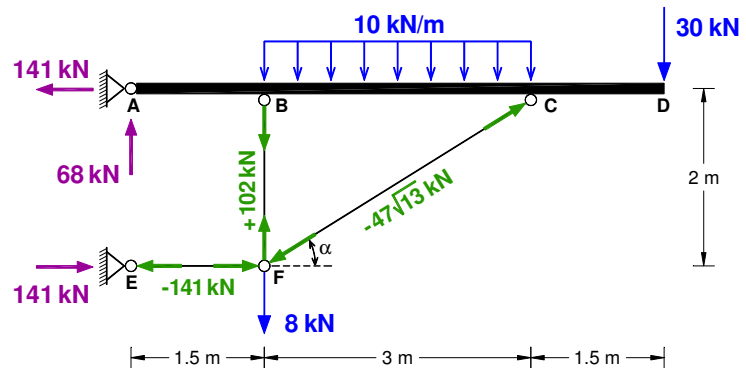


$$\tan \alpha = \frac{2}{3} \Rightarrow \alpha = \arctan \frac{2}{3} = 33,69^\circ \Rightarrow \begin{cases} \cos \alpha = \frac{3}{\sqrt{13}} \\ \sin \alpha = \frac{2}{\sqrt{13}} \end{cases}$$

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$$

$$\begin{cases} N_{FC} \cos \alpha + 141 = 0 \\ N_{FB} + N_{FC} \sin \alpha = 0 \end{cases}$$

$$\begin{cases} N_{FC} = -47\sqrt{13} \text{ kN (compressão)} \\ N_{FB} = 102 \text{ kN (tracção)} \end{cases}$$



• Características das barras

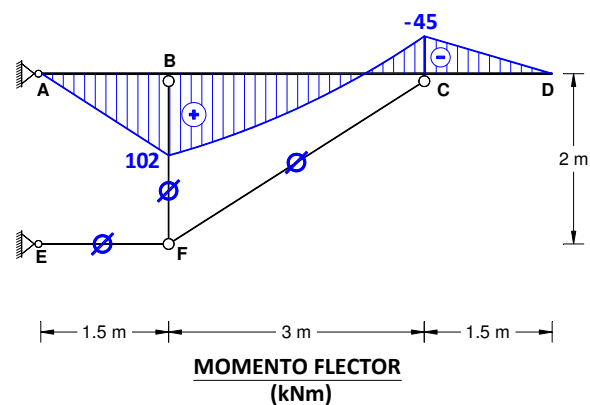
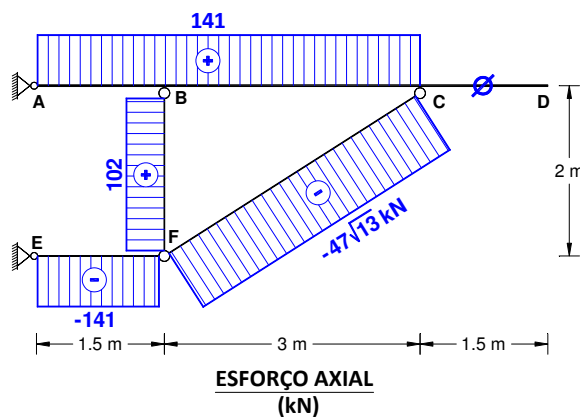
Barras **ABCD**

$$\begin{cases} E = 29 \text{ GPa} = 29 \times 10^6 \text{ kPa} \\ A = 0,40 \times 0,3 = 0,12 \text{ m}^2 \\ I = \frac{0,40 \times 0,30^3}{12} = 9 \times 10^{-4} \text{ m}^4 \end{cases}$$

Barras bi-articuladas **EF, FB e FC**

$$\begin{cases} E = 206 \text{ GPa} = 206 \times 10^6 \text{ kPa} \\ A = 0,10 \times 0,10 - 0,09 \times 0,09 = 1,9 \times 10^{-3} \text{ m}^2 \end{cases}$$

• Diagramas de Esforços – sistema real



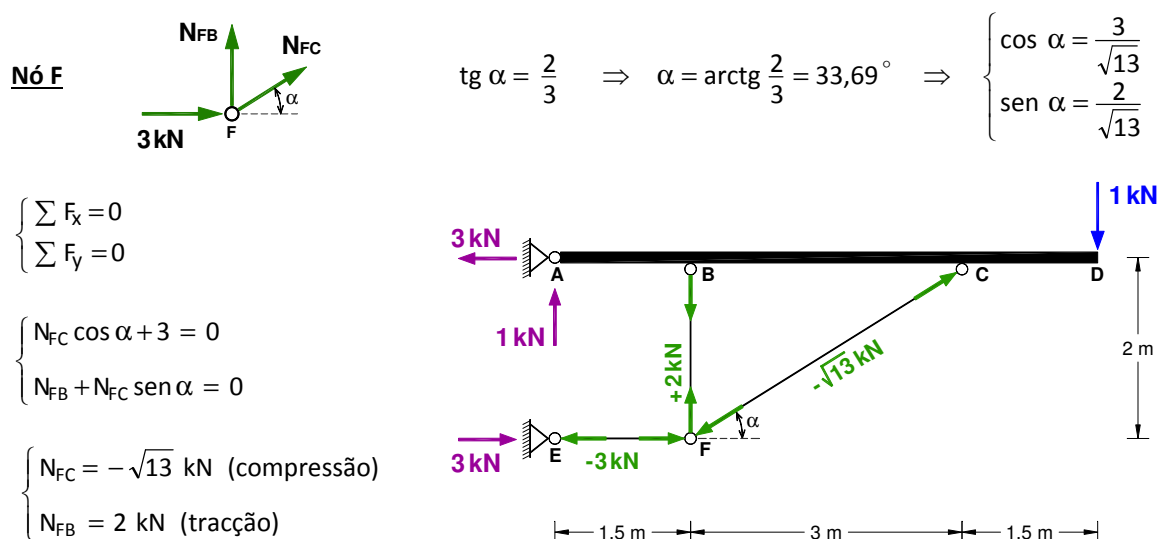
Barra BC $\Rightarrow M(z) = 102 - 34 \cdot z - 5z^2 \text{ kNm}$

a) DESLOCAMENTO DO PONTO D

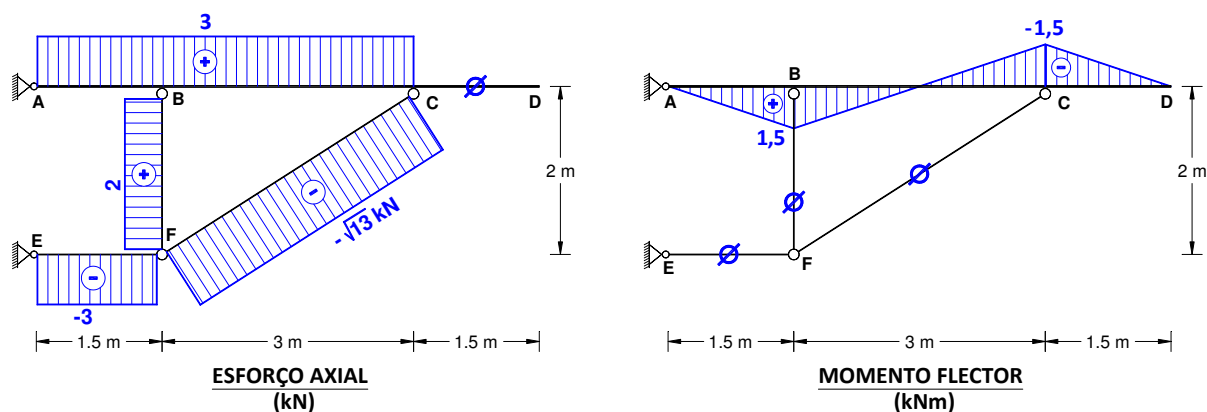
a₁) Deslocamento vertical do ponto D

$$\begin{cases} \sum F_x = 0 \Rightarrow H_A + H_E = 0 \\ \sum F_y = 0 \Rightarrow V_A - 1 = 0 \\ \sum M_A = 0 \Rightarrow H_E \times 2 - 1 \times 6 = 0 \end{cases} \Rightarrow \begin{cases} H_A = -3 \text{ kN} \leftarrow \\ V_A = 1 \text{ kN} \uparrow \\ H_E = 3 \text{ kN} \rightarrow \end{cases}$$

Nó E Barra EF $\Rightarrow N_{EF} = -3 \text{ kN}$ (compressão)



• Diagramas de Esforços – sistema virtual



Barra BC $\Rightarrow M(z) = 1,5 - z \text{ kNm}$

• **Aplicação do Teorema dos Trabalhos Virtuais (TTV)**

$$\bar{1} \times \delta_D^v = \underbrace{\int \frac{\bar{N} \times N}{EA} dz}_{\text{barras contínuas}} + \underbrace{\int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}}_{\text{barras bi-articuladas}}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \frac{1}{29 \times 10^6 \times 0,12} \times 3 \times 4,5 \times 141 = 5,4698 \times 10^{-4} \text{ m}$$

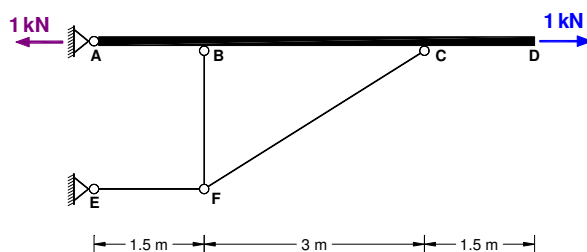
$$\begin{aligned} \int \frac{\bar{M} \times M}{EI} dz &= \frac{1}{29 \times 10^6 \times 9 \times 10^{-4}} \left[\frac{102 \times 1,5}{3} \times 1,5 + \int_0^3 \left[(1,5 - z) \times (102 - 34 \cdot z - 5z^2) \right] dz + \frac{-45 \times (-1,5) \times 1,5}{3} \right] = \\ &= \frac{1}{26100} \left[76,5 + \int_0^3 (5z^3 + 26,5z^2 - 153 \cdot z + 153) dz + 33,75 \right] = \\ &= \frac{1}{26100} \left(76,5 + \frac{5}{4} 3^4 + \frac{26,5}{3} 3^3 - \frac{153}{2} 3^2 + 153 \times 3 + 33,75 \right) = 8,448 \times 10^{-3} \text{ m} \end{aligned}$$

$$\sum \bar{N} \frac{N L}{EA} = \frac{1}{206 \times 10^6 \times 1,9 \times 10^{-3}} \left[-141 \times (-3) \times 1,5 + 102 \times 2 \times 2 - 47\sqrt{13} \times (-\sqrt{13}) \times \sqrt{13} \right] = 8,292 \times 10^{-3} \text{ m}$$

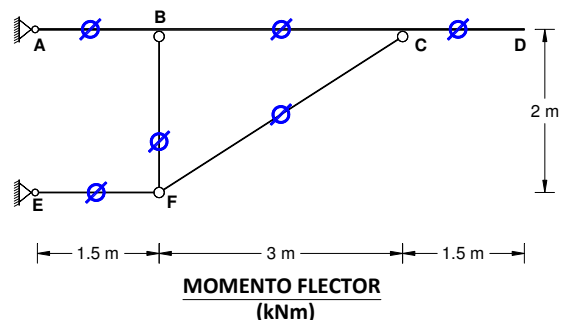
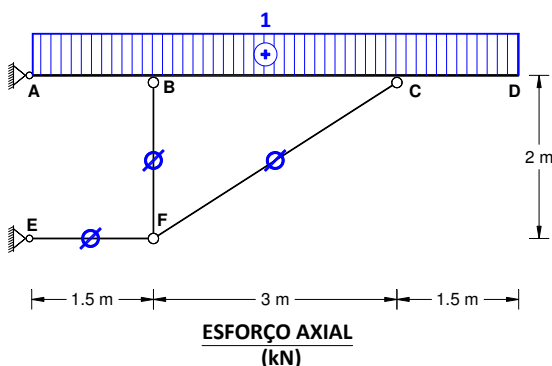
$$\delta_D^v = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}$$

$$\delta_D^v = 5,4698 \times 10^{-4} + 8,448 \times 10^{-3} + 8,292 \times 10^{-3} = 1,729 \times 10^{-2} \text{ m} = 1,729 \text{ cm} \downarrow$$

a2) Deslocamento horizontal do ponto D



• **Diagramas de Esforços – sistema virtual**



• Aplicação do Teorema dos Trabalhos Virtuais (TTV)

$$\bar{1} \times \delta_D^h = \underbrace{\int \frac{\bar{N} \times N}{EA} dz}_{\text{barras contínuas}} + \underbrace{\int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}}_{\text{barras bi-articuladas}}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \frac{1}{29 \times 10^6 \times 0,12} \times 1 \times 4,5 \times 141 = 1,823 \times 10^{-4} \text{ m} = 0,18 \text{ mm} \rightarrow$$

$$\int \frac{\bar{M} \times M}{EI} dz = 0$$

$$\sum \bar{N} \frac{N L}{EA} = 0$$

$$\delta_D^h = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA} = 0,18 \text{ mm} \rightarrow$$

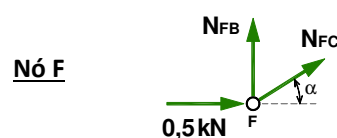
• Deslocamento do ponto D

$$\delta_D = \sqrt{(\delta_D^v)^2 + (\delta_D^h)^2} = \sqrt{1,729^2 + 0,018^2} = 1,729 \text{ cm} \rightarrow$$

a₃) Rotação do ponto D

$$\begin{cases} \sum F_x = 0 \Rightarrow H_A + H_E = 0 \\ \sum F_y = 0 \Rightarrow V_A = 0 \\ \sum M_A = 0 \Rightarrow H_E \times 2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} H_A = -0,5 \text{ kN} \leftarrow \\ V_A = 0 \\ H_E = 0,5 \text{ kN} \rightarrow \end{cases}$$

Nó E Barra EF $\Rightarrow N_{EF} = -0,5 \text{ kN}$ (compressão)

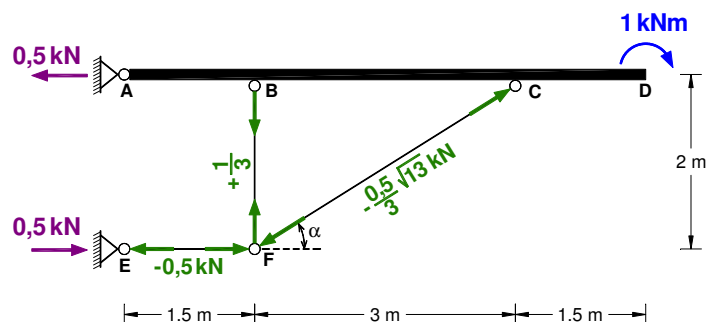


$$\tan \alpha = \frac{2}{3} \Rightarrow \alpha = \arctg \frac{2}{3} = 33,69^\circ \Rightarrow \begin{cases} \cos \alpha = \frac{3}{\sqrt{13}} \\ \sin \alpha = \frac{2}{\sqrt{13}} \end{cases}$$

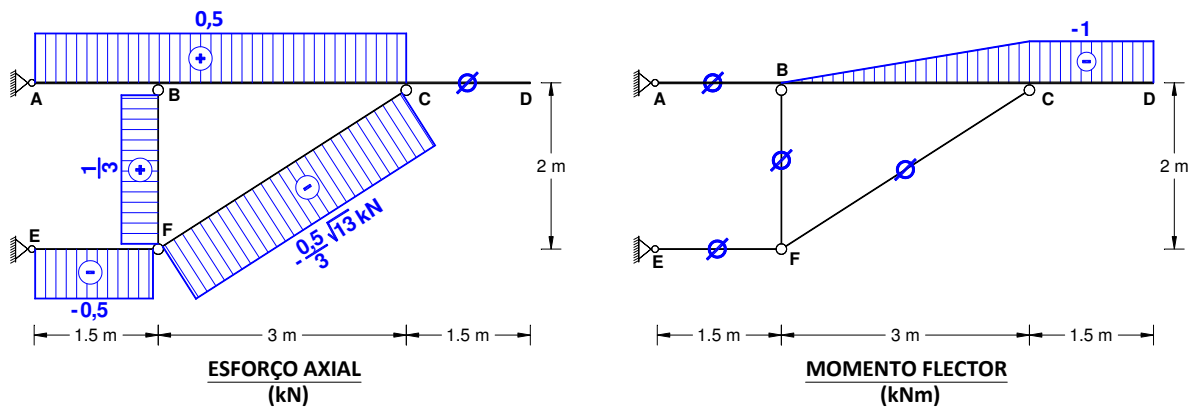
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$$

$$\begin{cases} N_{FC} \cos \alpha + 0,5 = 0 \\ N_{FB} + N_{FC} \sin \alpha = 0 \end{cases}$$

$$\begin{cases} N_{FC} = -\frac{0,5}{3} \sqrt{13} \text{ kN (compressão)} \\ N_{FB} = \frac{1}{3} \text{ kN (tracção)} \end{cases}$$



• Diagramas de Esforços – sistema virtual



Barra BC $\Rightarrow M(z) = -\frac{1}{3}z \text{ kNm}$

• Aplicação do Teorema dos Trabalhos Virtuais (TTV)

$$\bar{1} \times \theta_D = \underbrace{\int \frac{\bar{N} \times N}{EA} dz}_{\text{barras contínuas}} + \underbrace{\int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}}_{\text{barras bi-articuladas}}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \frac{1}{29 \times 10^6 \times 0,12} \times 0,5 \times 4,5 \times 141 = 9,116 \times 10^{-5} \text{ rad}$$

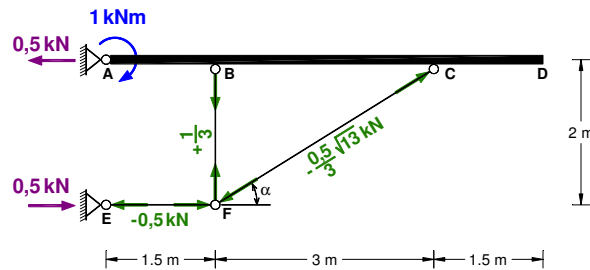
$$\begin{aligned} \int \frac{\bar{M} \times M}{EI} dz &= \frac{1}{29 \times 10^6 \times 9 \times 10^{-4}} \left[\int_0^3 \left[\left(-\frac{1}{3}z\right) \times (102 - 34 \cdot z - 5z^2) \right] dz + \frac{-45 \times 1,5 \times (-1)}{2} \right] = \\ &= \frac{1}{26100} \left[\int_0^3 \left(\frac{5}{3}z^3 + \frac{34}{3}z^2 - 34 \cdot z \right) dz + 33,75 \right] = \\ &= \frac{1}{26100} \left(\frac{5}{12}3^4 + \frac{34}{9}3^3 - \frac{34}{2} \times 3^2 - 33,75 \right) = 6,284 \times 10^{-4} \text{ rad} \end{aligned}$$

$$\begin{aligned} \sum \bar{N} \frac{N L}{EA} &= \frac{1}{206 \times 10^6 \times 1,9 \times 10^{-3}} \left[-141 \times (-0,5) \times 1,5 + 102 \times \frac{1}{3} \times 2 - 47\sqrt{13} \times \left(-\frac{0,5}{3}\sqrt{13}\right) \times \sqrt{13} \right] = \\ &= 1,382 \times 10^{-3} \text{ rad} \end{aligned}$$

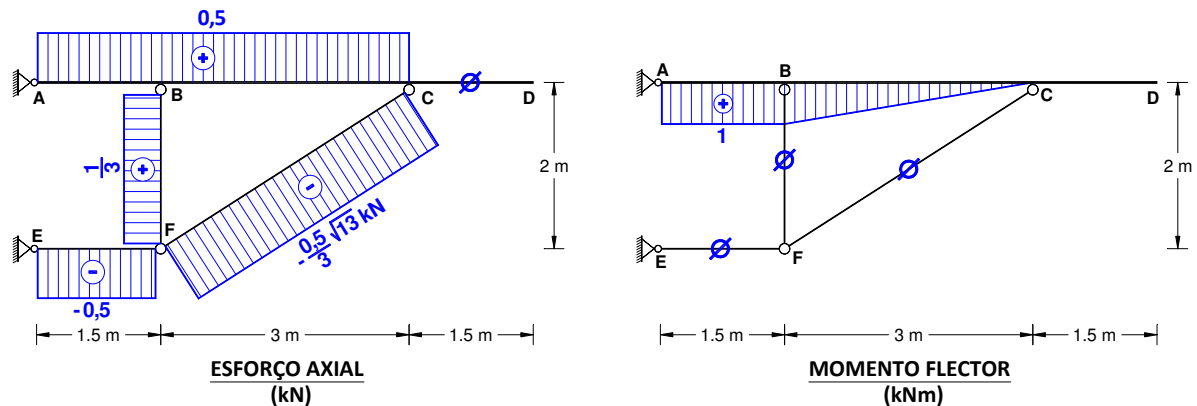
$$\theta_D = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}$$

$$\theta_D = 9,116 \times 10^{-5} + 6,284 \times 10^{-4} + 1,382 \times 10^{-3} = 2,102 \times 10^{-3} \text{ rad } \circlearrowleft$$

b) ROTAÇÃO DO PONTO A



• Diagramas de Esforços – sistema virtual



Barra BC $\Rightarrow M(z) = 1 - \frac{1}{3}z$ kNm

• Aplicação do TTV

$$\bar{1} \times \theta_A = \underbrace{\int \frac{\bar{N} \times N}{EA} dz}_{\text{barras contínuas}} + \underbrace{\int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}}_{\text{barras bi-articuladas}}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \frac{1}{29 \times 10^6 \times 0,12} \times 0,5 \times 4,5 \times 141 = 9,116 \times 10^{-5} \text{ rad}$$

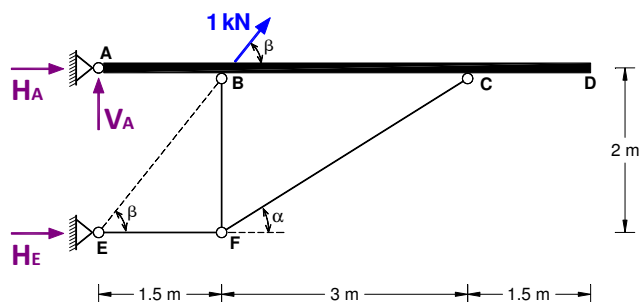
$$\begin{aligned} \int \frac{\bar{M} \times M}{EI} dz &= \frac{1}{29 \times 10^6 \times 9 \times 10^{-4}} \left[\frac{102 \times 1,5 \times 1}{2} + \int_0^3 \left(1 - \frac{1}{3}z \right) \times (102 - 34 \cdot z - 5z^2) dz \right] = \\ &= \frac{1}{26100} \left[76,5 + \int_0^3 \left(\frac{5}{3}z^3 + \frac{19}{3}z^2 - 68 \cdot z + 102 \right) dz \right] = \\ &= \frac{1}{26100} \left(76,5 + \frac{5}{12}3^4 + \frac{19}{9}3^3 - \frac{68}{2} \times 3^2 + 102 \times 3 \right) = 6,408 \times 10^{-3} \text{ rad} \end{aligned}$$

$$\begin{aligned} \sum \bar{N} \frac{N L}{EA} &= \frac{1}{206 \times 10^6 \times 1,9 \times 10^{-3}} \left[-141 \times (-0,5) \times 1,5 + 102 \times \frac{1}{3} \times 2 - 47\sqrt{13} \times \left(-\frac{0,5}{3}\sqrt{13} \right) \times \sqrt{13} \right] = \\ &= 1,382 \times 10^{-3} \text{ rad} \end{aligned}$$

$$\theta_A = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA} = 9,116 \times 10^{-5} + 6,408 \times 10^{-3} + 1,382 \times 10^{-3} = 7,881 \times 10^{-3} \text{ rad } \curvearrowright$$

c) DESLOCAMENTO RELATIVO E-B

Como o ponto **E** está fixo (apoio sem assentamento), o deslocamento relativo dos pontos **E** e **B** (δ_{E-B}) será igual ao deslocamento do ponto **B** na direcção $\overrightarrow{EB}$ ($\delta_{B-\text{dir } \overrightarrow{EB}}$).

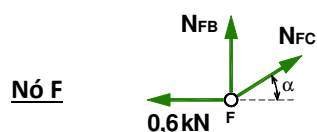


$$\tan \beta = \frac{2}{1,5} \Rightarrow \beta = \arctan \frac{2}{1,5} = 53,13^\circ$$

$$\beta \begin{cases} \cos \beta = 0,6 \\ \sin \beta = 0,8 \end{cases}$$

$$\begin{cases} \sum F_x = 0 \Rightarrow H_A + H_E + 1 \times \cos \beta = 0 \\ \sum F_y = 0 \Rightarrow V_A + 1 \times \sin \beta = 0 \\ \sum M_A = 0 \Rightarrow H_E \times 2 + 1 \times \sin \beta \times 1,5 = 0 \end{cases} \Rightarrow \begin{cases} H_A = 0 \\ V_A = -0,8 \text{ kN} \downarrow \\ H_E = -0,6 \text{ kN} \leftarrow \end{cases}$$

Nó E Barra EF $\Rightarrow N_{EF} = 0,6 \text{ kN}$ (tracção)

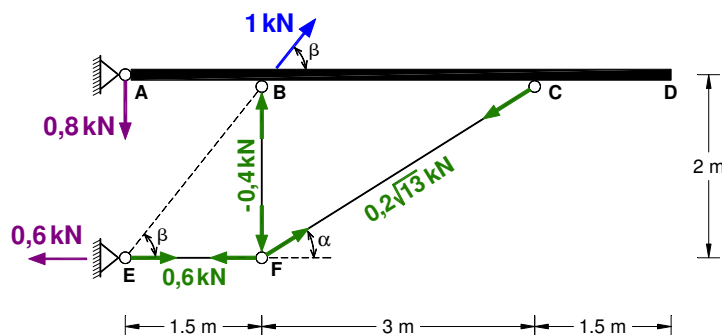


$$\tan \alpha = \frac{2}{3} \Rightarrow \alpha = \arctan \frac{2}{3} = 33,69^\circ \Rightarrow \begin{cases} \cos \alpha = \frac{3}{\sqrt{13}} \\ \sin \alpha = \frac{2}{\sqrt{13}} \end{cases}$$

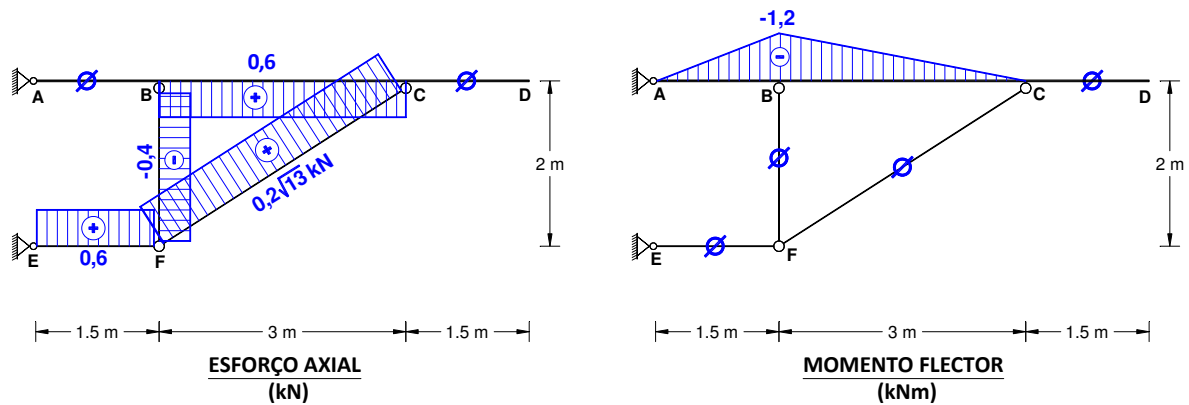
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$$

$$\begin{cases} N_{FC} \cos \alpha - 0,6 = 0 \\ N_{FB} + N_{FC} \sin \alpha = 0 \end{cases}$$

$$\begin{cases} N_{FC} = 0,2\sqrt{13} \text{ kN (tracção)} \\ N_{FB} = -0,4 \text{ kN (compressão)} \end{cases}$$



• Diagramas de Esforços – sistema virtual



Barra BC $\Rightarrow M(z) = -1,2 + 0,4z$ kNm

• Aplicação do Teorema dos Trabalhos Virtuais (TTV)

$$\bar{1} \times \delta_{B-\text{dir } \overline{EB}} = \underbrace{\int \frac{\bar{N} \times N}{EA} dz}_{\text{barras contínuas}} + \underbrace{\int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}}_{\text{barras bi-articuladas}}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \frac{1}{29 \times 10^6 \times 0,12} \times 0,6 \times 3 \times 141 = 7,293 \times 10^{-5} \text{ m}$$

$$\begin{aligned} \int \frac{\bar{M} \times M}{EI} dz &= \frac{1}{29 \times 10^6 \times 9 \times 10^{-4}} \left[\frac{102 \times (-1,2) \times 1,5}{3} + \int_0^3 [(-1,2 + 0,4z) \times (102 - 34z - 5z^2)] dz \right] = \\ &= \frac{1}{26100} \left[-61,2 + \int_0^3 (-2z^3 - 7,6z^2 + 81,6z - 122,4) dz \right] = \\ &= \frac{1}{26100} \left(-61,2 - \frac{2}{4} 3^4 - \frac{7,6}{3} 3^3 + \frac{81,6}{2} \times 3^2 - 122,4 \times 3 \right) = -6,517 \times 10^{-3} \text{ m} \end{aligned}$$

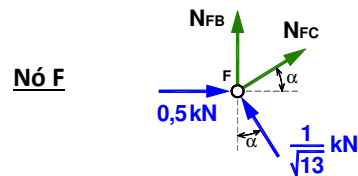
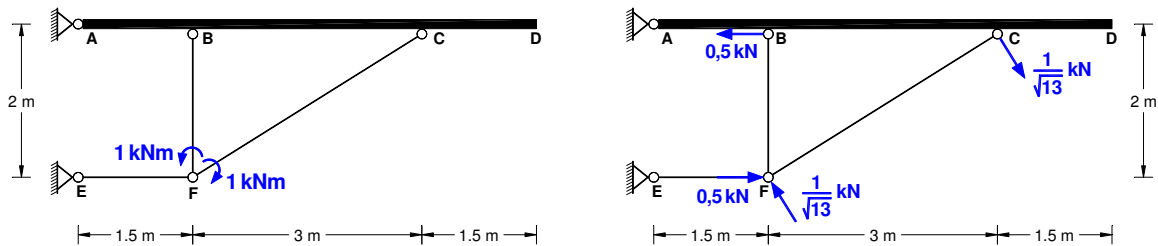
$$\begin{aligned} \sum \bar{N} \frac{N L}{EA} &= \frac{1}{206 \times 10^6 \times 1,9 \times 10^{-3}} \left[-141 \times 0,6 \times 1,5 + 102 \times (-0,4) \times 2 - 47\sqrt{13} \times 0,2\sqrt{13} \times \sqrt{13} \right] = \\ &= -1,658 \times 10^{-3} \text{ m} \end{aligned}$$

$$\delta_{B-\text{dir } \overline{EB}} = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}$$

$$\delta_{B-\text{dir } \overline{EB}} = 7,293 \times 10^{-5} - 6,517 \times 10^{-3} - 1,658 \times 10^{-3} = -8,102 \times 10^{-3} \text{ m}$$

$$\delta_{E-B} = \delta_{B-\text{dir } \overline{EB}} = -8,102 \text{ mm} \quad (\text{os pontos E e B aproximaram-se})$$

d) VARIAÇÃO DO ÂNGULO BFC



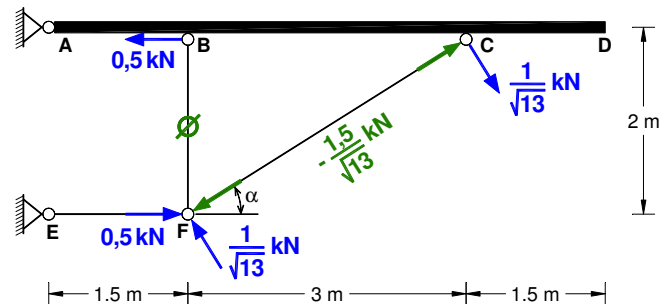
$$\operatorname{tg} \alpha = \frac{2}{3} \Rightarrow \alpha = \arctg \frac{2}{3} = 33,69^\circ$$

$$\Rightarrow \begin{cases} \cos \alpha = \frac{3}{\sqrt{13}} \\ \sin \alpha = \frac{2}{\sqrt{13}} \end{cases}$$

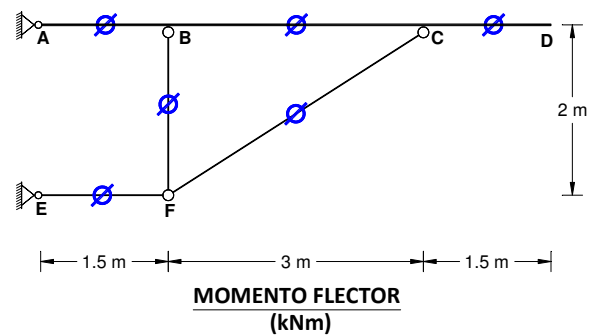
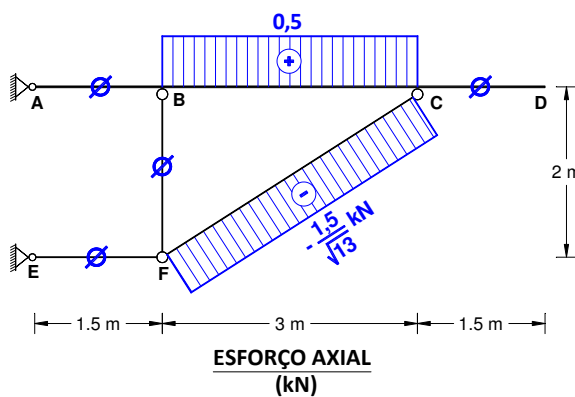
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \end{cases}$$

$$\begin{cases} N_{FC} \cos \alpha + 0,5 - \frac{1}{\sqrt{13}} \sin \alpha = 0 \\ N_{FB} + N_{FC} \sin \alpha + \frac{1}{\sqrt{13}} \cos \alpha = 0 \end{cases}$$

$$\begin{cases} N_{FC} = -\frac{1,5}{\sqrt{13}} \text{ kN (compressão)} \\ N_{FB} = 0 \end{cases}$$



• Diagramas de Esforços – sistema virtual



• Aplicação do Teorema dos Trabalhos Virtuais (TTV)

$$\bar{1} \times \Delta\varphi_{BFC} = \underbrace{\int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz}_{\text{barras contínuas}} + \underbrace{\sum \bar{N} \frac{N L}{EA}}_{\text{barras bi-articuladas}}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \frac{1}{29 \times 10^6 \times 0,12} \times 0,5 \times 3 \times 141 = 6,078 \times 10^{-5} \text{ rad}$$

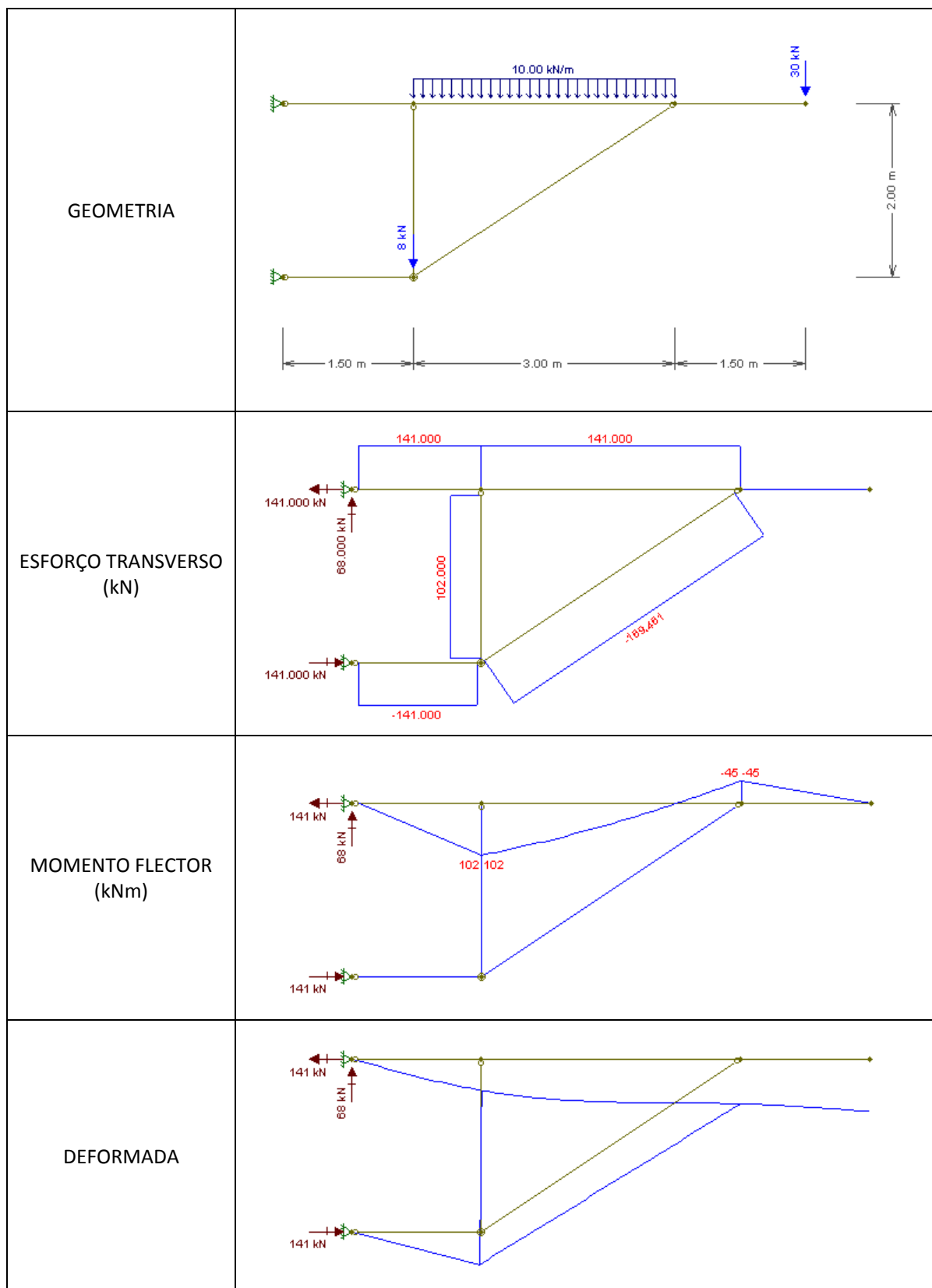
$$\int \frac{\bar{M} \times M}{EI} dz = 0$$

$$\sum \bar{N} \frac{N L}{EA} = \frac{-47\sqrt{13} \times (-\frac{1,5}{\sqrt{13}}) \times \sqrt{13}}{206 \times 10^6 \times 1,9 \times 10^{-3}} = 6,494 \times 10^{-4} \text{ rad}$$

$$\Delta\varphi_{BFC} = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \sum \bar{N} \frac{N L}{EA}$$

$$\Delta\varphi_{BFC} = 6,078 \times 10^{-5} + 6,494 \times 10^{-4} = 7,102 \times 10^{-4} \text{ rad} \quad (\text{o ângulo BFC aumentou})$$

e) PROGRAMA FTOOL



Deslocamentos e rotações obtidos com o programa FTOOL

<u>Deslocamento e rotação do ponto D</u> Sentidos positivos: 	Node Results Nodal Displacements: Dx = 1.823e-002 cm Dy = -1.729e+000 cm Rz = -2.105e-003 rad
<u>Rotação do ponto A</u> Sentidos positivos: 	Member Displacements and Rotations Init: Dx: 0.000e+000 cm Dy: 0.000e+000 cm Rz: -7.881e-003 rad End: Dx: 6.078e-003 cm Dy: -1.036e+000 cm Rz: -4.950e-003 rad