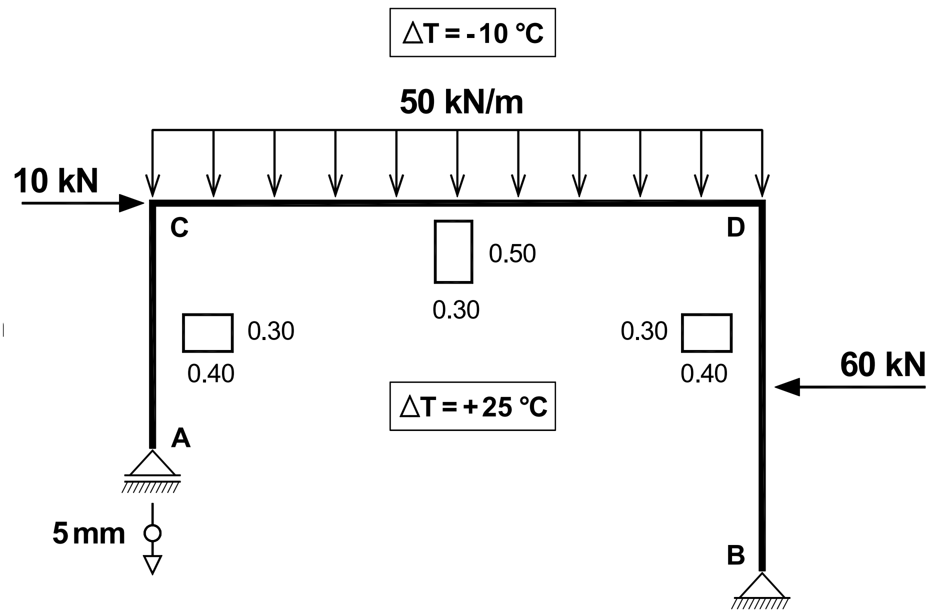
Instituto Superior de Engenharia do Porto

LICENCIATURA EM ENGENHARIA CIVIL

TEORIA DE ESTRUTURAS

TEOREMA DOS TRABALHOS VIRTUAIS



ESTRUTURA CONTÍNUA ISOSTÁTICA

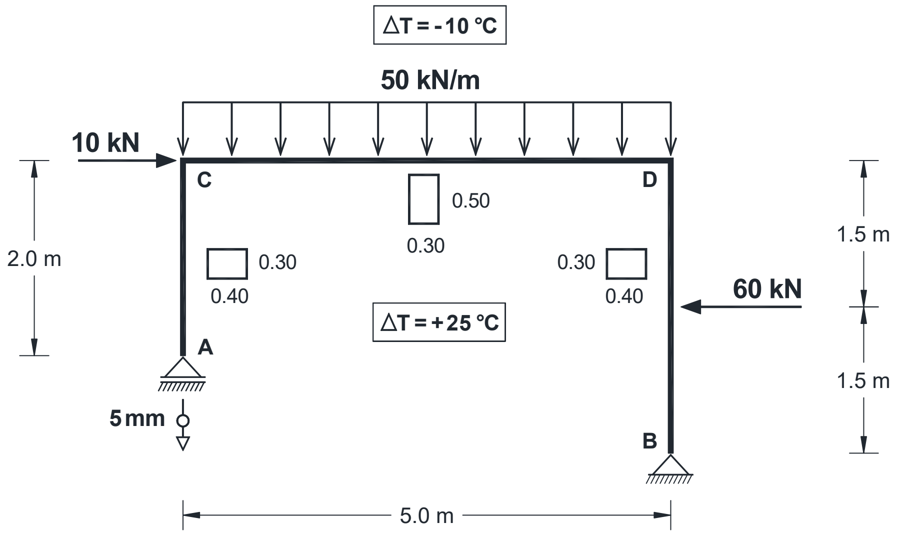
ISABEL ALVIM TELES

EXERCÍCIO

Considere a estrutura de betão (E=30 GPa; α=1,2x10^{-5} /°C) constituída por dois pilares e uma viga com secções rectangulares conforme representado na figura.

Para além do carregamento, a estrutura está submetida à seguinte variação de temperatura: -10 °C no exterior e +25°C no interior.

O apoio **A** sofreu um assentamento de 5 mm.



Responda às alíneas seguintes desprezando a contribuição do esforço transverso.

a) Determine o deslocamento do apoio **A**;

b) Determine a rotação do pilar no apoio **B**;

c) Determine que força horizontal deverá ser aplicada em **D** para que o apoio **A** não sofra qualquer deslocamento horizontal.

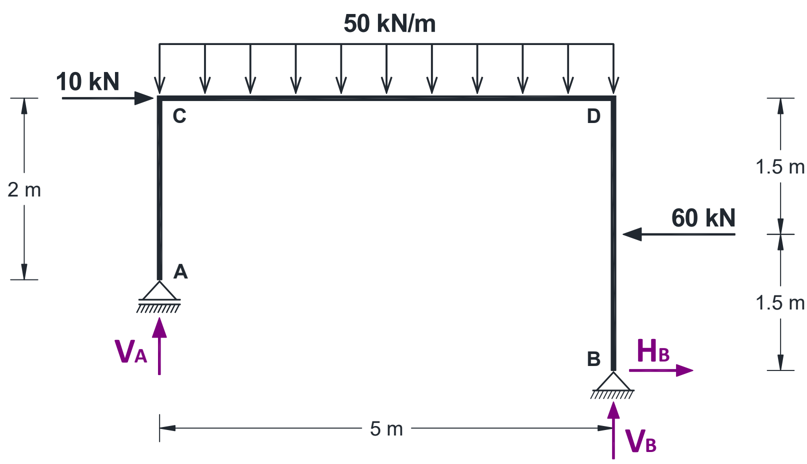
RESOLUÇÃO

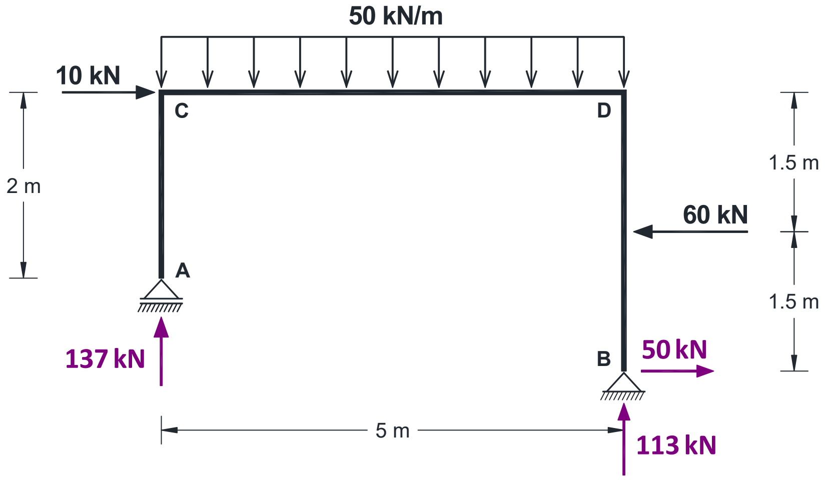
- **Cálculo das reacções – sistema real**

$$\begin{cases} \sum Fx = 0 \\ \sum Fy = 0 \\ \sum M_B = 0 \end{cases}$$

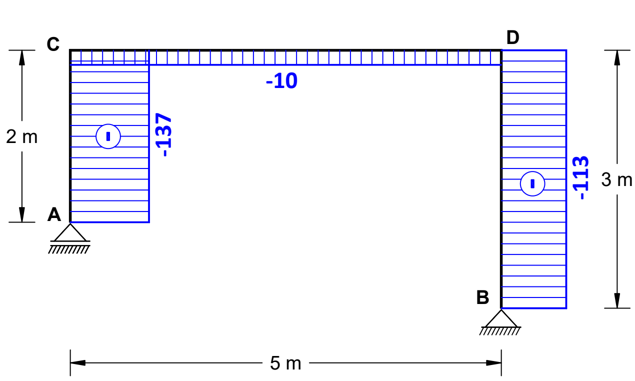
$$\begin{cases} H_B + 10 - 60 = 0 \\ V_A + V_B - 50\times 5 = 0 \\ 10\times 3 - 50\times 5\times 2{,}5 - 60\times 1{,}5 + 5\ V_A = 0 \end{cases}$$

$$\begin{cases} H_B = 50\ \text{kN}\ \rightarrow \\ V_B = 113\ \text{kN}\ \uparrow \\ V_A = 137\ \text{kN}\ \uparrow \end{cases}$$

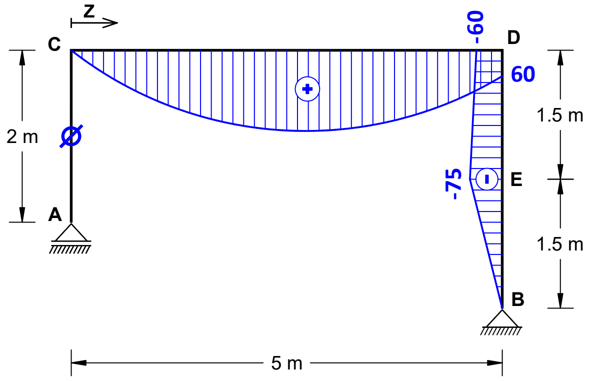




- **Diagramas de Esforços – sistema real**



Esforço axial **N** (kN)



Momento flector **M** (kNm)

Barra **CD** – expressão do momento flector: $M(z) = 137 \times \mathbf{Z} - 25 \times \mathbf{Z}^2$

Alínea a) - Deslocamento do apoio A

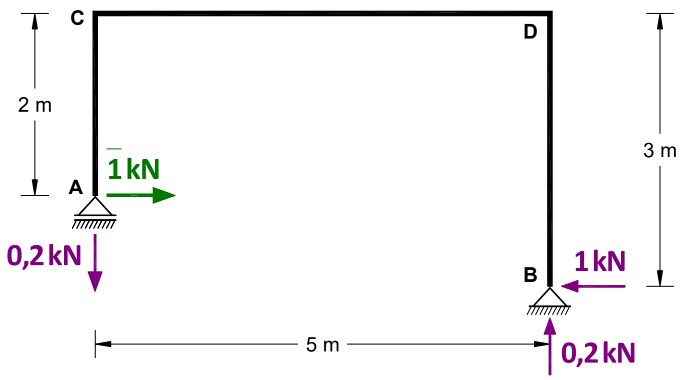
O deslocamento vertical do apoio **A** corresponde ao seu assentamento, ou seja: $\delta_A^V = 5 \text{ mm} \downarrow$

Passemos ao cálculo do deslocamento horizontal do apoio **A**.

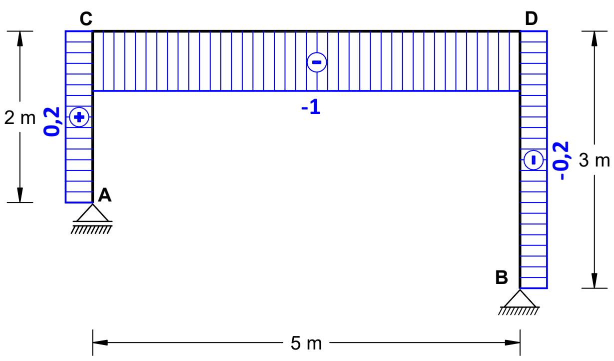
- **Deslocamento horizontal de A – sistema virtual**

$$\begin{cases} \sum Fx = 0 \\ \sum Fy = 0 \\ \sum M = 0 \end{cases}$$

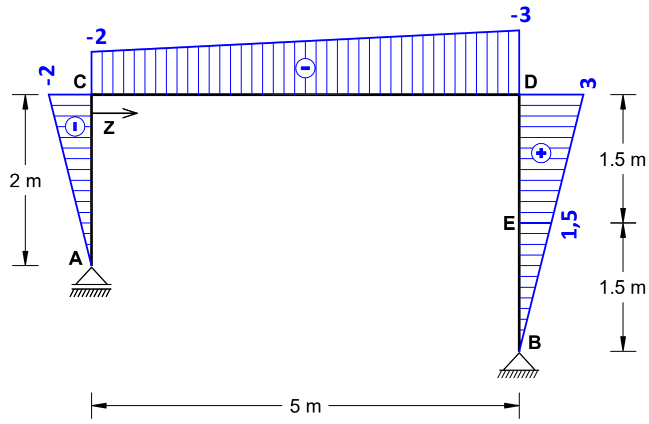
$$\begin{cases} H_B = 1 \text{ kN} \leftarrow \\ V_B = 0{,}2 \text{ kN} \uparrow \\ V_A = 0{,}2 \text{ kN} \downarrow \end{cases}$$



- **Diagramas de Esforços – sistema virtual**



Esforço axial $\overline{\mathbf{N}}$ (kN)



Momento flector $\overline{\mathbf{M}}$ (kNm)

Barra **CD** – expressão do momento flector: $\overline{M}(z) = 2 + 0{,}2 \times \mathbf{z}$

PILARES **Barras AC e BD** (bxh) = 0,30x0,40 m²	$EA = 30\times10^6 \times 0{,}30\times0{,}40 = 3{,}6\times10^6 \ \text{kPa}\times\text{m}^2$ $EI = 30\times10^6 \times \dfrac{0{,}30\times0{,}40^3}{12} = 4{,}8\times10^4 \ \text{kPa}\times\text{m}^4$
VIGA **Barra CD** (bxh) = 0,30x0,5 m²	$EA = 30\times10^6 \times 0{,}30\times0{,}50 = 4{,}5\times10^6 \ \text{kPa}\times\text{m}^2$ $EI = 30\times10^6 \times \dfrac{0{,}30\times0{,}50^3}{12} = 9{,}375\times10^4 \ \text{kPa}\times\text{m}^4$

TEOREMA DOS TRABALHOS VIRTUAIS (desprezando a contribuição do esforço transverso):

$$\boxed{\overline{1} \times \delta_A^h + \Sigma\left(\overline{R} \times \text{assent. apoio}\right) = \int \frac{\overline{N}\times N}{EA}dz + \int \frac{\overline{M}\times M}{EI}dz + \int \overline{N}\,.\,\alpha\,.\,T_m\,.dz + \int \overline{M}\,.\,\alpha\,\frac{T_{inf} - T_{sup}}{h}.dz}$$

$$\int \frac{\overline{N}\times N}{EA}dz = \int_A^C \frac{\overline{N}\times N}{EA}dz + \int_B^D \frac{\overline{N}\times N}{EA}dz + \int_C^D \frac{\overline{N}\times N}{EA}dz =$$

$$= \frac{1}{3{,}6\times10^6}\left[0{,}2\times(-137)\times2 + (-0{,}2)\times(-113)\times3\right] + \frac{1}{4{,}5\times10^6}\left[(-1)\times(-10)\times5\right] = 14{,}722\times10^{-6}$$

$$\int \frac{\overline{M}\times M}{EI}dz = \left[\int \frac{\overline{M}\times M}{EI}dz\right]_{\text{Pilares}} + \left[\int \frac{\overline{M}\times M}{EI}dz\right]_{\text{Viga}} = -3{,}5156\times10^{-3} - 18{,}1556\times10^{-3} = -21{,}6712\times10^{-3}$$

$$\left[\int \frac{\overline{M}\times M}{EI}dz\right]_{\text{Pilares}} = \frac{1}{4{,}8\times10^4}\left[\frac{1{,}5\times(-75)\times1{,}5}{3} + \frac{1{,}5}{6}\left(2\times1{,}5\times(-75) + 1{,}5\times(-60) + 3\times(-75) + 2\times3\times(-60)\right)\right] =$$

$$= \frac{1}{4{,}8\times10^4}(56{,}25 - 225) = -3{,}5156\times10^{-3}$$

$$\left[\int \frac{\overline{M}\times M}{EI}dz\right]_{Viga} = \frac{1}{9{,}375\times10^{4}}\int_0^6 (2+0{,}2\times Z)\times(137\times Z-25\times Z^2)\,dz =$$

$$= \frac{1}{9{,}375\times10^{4}}\int_0^6 (5\times Z^3+22{,}6\times Z^2-274\times Z)\,dz =$$

$$= \frac{1}{9{,}375\times10^{4}}\left[\frac{5}{4}Z^4+\frac{22{,}6}{3}Z^3-\frac{274}{2}Z^2\right]_0^5 = -18{,}1556\times10^{-3}$$

$$T_m = \frac{-10+25}{2} = 7{,}5\,^{o}C$$

$$\int \overline{N}.\alpha.T_m.dz = \alpha.T_m.\int \overline{N}.dz = 1{,}2\times10^{-5}\times7{,}5\times\int \overline{N}.dz =$$

$$= 1{,}2\times10^{-5}\times7{,}5\times(0{,}2\times2-1\times5-0{,}2\times3) = -4{,}68\times10^{-4}$$

Barra **AC** (h=0,40 m)	$\begin{cases} T_{inf}=25\,^oC \\ T_{sup}=-10\,^oC \end{cases}$	$\Rightarrow \dfrac{T_{inf}-T_{sup}}{h} = \dfrac{25-(-10)}{0{,}40} = 87{,}5$
Barra **CD** (h=0,50 m)	$\begin{cases} T_{inf}=25\,^oC \\ T_{sup}=-10\,^oC \end{cases}$	$\Rightarrow \dfrac{T_{inf}-T_{sup}}{h} = \dfrac{25-(-10)}{0{,}50} = 70$
Barra **BD** (h=0,40 m)	$\begin{cases} T_{inf}=-10\,^oC \\ T_{sup}=25\,^oC \end{cases}$	$\Rightarrow \dfrac{T_{inf}-T_{sup}}{h} = \dfrac{(-10)-25}{0{,}40} = -87{,}5$

$$\int \overline{M}.\alpha\frac{T_{inf}-T_{sup}}{h}.dz = 1{,}2\times10^{-5}\times87{,}5\times\left[\int\overline{M}\,dz\right]_A^C + 1{,}2\times10^{-5}\times70\times\left[\int\overline{M}\,dz\right]_C^D + 1{,}2\times10^{-5}\times(-87{,}5)\times\left[\int\overline{M}\,dz\right]_B^D =$$

$$= 1{,}05\times10^{-3}\times\frac{-2\times2}{2} + 0{,}84\times10^{-3}\times\frac{-2+(-3)}{2}\times5 - 1{,}05\times10^{-3}\times\frac{3\times3}{2} =$$

$$= (-2{,}1-10{,}5-4{,}725)\times10^{-3} = -17{,}325\times10^{-3}$$

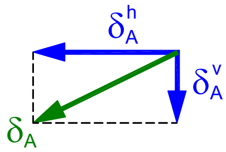
$$\overline{1}\times\delta_A^h + \Sigma\left(\overline{R}\times\text{assent. apoio}\right) = \int\frac{\overline{N}\times N}{EA}dz + \int\frac{\overline{M}\times M}{EI}dz + \int\overline{N}.\alpha.T_m.dz + \int\overline{M}.\alpha\frac{T_{inf}-T_{sup}}{h}.dz$$

$$\delta_A^h + 0{,}2\times0{,}005 = 14{,}722\times10^{-6} - 21{,}6712\times10^{-3} - 4{,}68\times10^{-4} - 17{,}325\times10^{-3}$$

$$\delta_A^h = -4{,}04\times10^{-2}m = -4{,}04\ cm = -40{,}4\ mm \ \Leftarrow$$

$$\delta_A^v = 5\ mm \ \Downarrow$$

$$\delta_A = \sqrt{(\delta_A^h)^2+(\delta_A^v)^2} = \sqrt{(40{,}4^2+5^2} = 40{,}7\ cm \ \swarrow$$

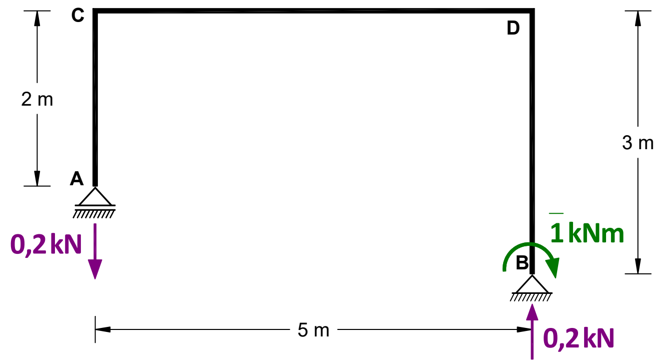


Alínea b) - Rotação do apoio B

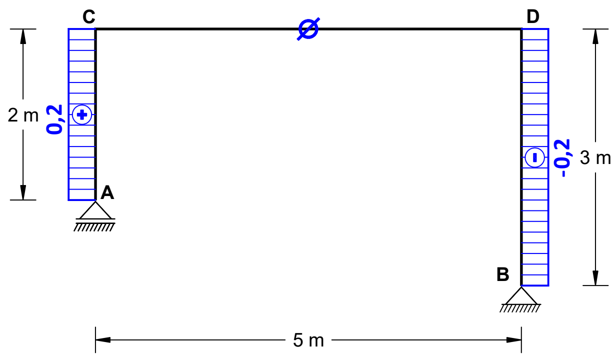
- **Rotação de B – sistema virtual**

$$\begin{cases} \sum Fx = 0 \\ \sum Fy = 0 \\ \sum M = 0 \end{cases}$$

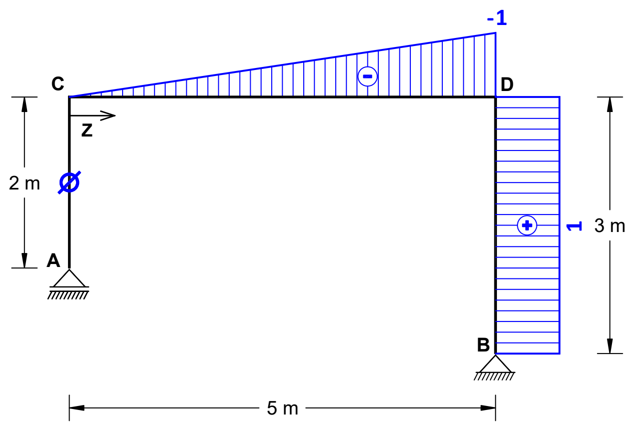
$$\begin{cases} H_B = 0 \\ V_B = 0{,}2 \text{ kN} \uparrow \\ V_A = 0{,}2 \text{ kN} \downarrow \end{cases}$$



- **Diagramas de Esforços – sistema virtual**



Esforço axial $\bar{\mathbf{N}}$ (kN)



Momento flector $\bar{\mathbf{M}}$ (kNm)

Barra **CD** – expressão do momento flector: $\bar{M}(z) = -0{,}2 \times \mathbf{Z}$

TEOREMA DOS TRABALHOS VIRTUAIS (desprezando a contribuição do esforço transverso):

$$\boxed{\bar{1} \times \theta_B + \sum\left(\bar{R} \times \text{assent. apoio}\right) = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \int \bar{N}.\alpha.T_m.dz + \int \bar{M}.\alpha \frac{T_{inf} - T_{sup}}{h}.dz}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \int_A^C \frac{\bar{N} \times N}{EA} dz + \int_B^D \frac{\bar{N} \times N}{EA} dz + \int_C^D \frac{\bar{N} \times N}{EA} dz =$$

$$= \frac{1}{3{,}6 \times 10^6} \left[0{,}2 \times (-137) \times 2 + (-0{,}2) \times (-113) \times 3\right] + 0 = 3{,}611 \times 10^{-6}$$

$$\int \frac{\overline{M}\times M}{EI}dz = \left[\int \frac{\overline{M}\times M}{EI}dz\right]_{Pilares} + \left[\int \frac{\overline{M}\times M}{EI}dz\right]_{Viga} = -3{,}28125\times10^{-3} - 1{,}4311\times10^{-3} = -4{,}71235\times10^{-3}$$

$$\left[\int \frac{\overline{M}\times M}{EI}dz\right]_{Pilares} = \frac{1}{4{,}8\times10^{4}}\left[\frac{1\times(-75)\times1{,}5}{2} + \frac{-75-60}{2}\times1{,}5\times1\right] = \frac{-157{,}5}{4{,}8\times10^{4}} = -3{,}28125\times10^{-3}$$

$$\left[\int \frac{\overline{M}\times M}{EI}dz\right]_{Viga} = \frac{1}{9{,}375\times10^{4}}\int_0^6(-0{,}2\times Z)\times(137\times Z-25\times Z^2)\,dz =$$

$$= \frac{1}{9{,}375\times10^{4}}\int_0^6(5\times Z^2-27{,}4\times Z)\,dz =$$

$$= \frac{1}{9{,}375\times10^{4}}\left[\frac{5}{3}Z^3-\frac{27{,}4}{2}Z^2\right]_0^5 = -1{,}4311\times10^{-3}$$

$$T_m = \frac{-10+25}{2} = 7{,}5\ ^{o}C$$

$$\int \overline{N}.\alpha.T_m.dz = \alpha.T_m.\int \overline{N}.dz = 1{,}2\times10^{-5}\times7{,}5\times\int \overline{N}.dz =$$

$$= 1{,}2\times10^{-5}\times7{,}5\times(0{,}2\times2-0{,}2\times3) = -1{,}8\times10^{-5}$$

Barra **AC** (h=0,40 m)	$\begin{cases} T_{inf} = 25\ ^{o}C \\ T_{sup} = -10\ ^{o}C \end{cases}$	$\Rightarrow$	$\frac{T_{inf}-T_{sup}}{h} = \frac{25-(-10)}{0{,}40} = 87{,}5$
Barra **CD** (h=0,50 m)	$\begin{cases} T_{inf} = 25\ ^{o}C \\ T_{sup} = -10\ ^{o}C \end{cases}$	$\Rightarrow$	$\frac{T_{inf}-T_{sup}}{h} = \frac{25-(-10)}{0{,}50} = 70$
Barra **BD** (h=0,40 m)	$\begin{cases} T_{inf} = -10\ ^{o}C \\ T_{sup} = 25\ ^{o}C \end{cases}$	$\Rightarrow$	$\frac{T_{inf}-T_{sup}}{h} = \frac{(-10)-25}{0{,}40} = -87{,}5$

$$\int \overline{M}.\alpha\frac{T_{inf}-T_{sup}}{h}.dz = 1{,}2\times10^{-5}\times87{,}5\times\left[\int \overline{M}\,dz\right]_A^C + 1{,}2\times10^{-5}\times70\times\left[\int \overline{M}\,dz\right]_C^D + 1{,}2\times10^{-5}\times(-87{,}5)\times\left[\int \overline{M}\,dz\right]_B^D =$$

$$= 0 + 0{,}84\times10^{-3}\times\frac{-1\times5}{2} - 1{,}05\times10^{-3}\times(1\times3) = -5{,}25\times10^{-3}$$

$$\overline{1}\times\theta_B + \Sigma\left(\overline{R}\times \text{assent. apoio}\right) = \int\frac{\overline{N}\times N}{EA}dz + \int\frac{\overline{M}\times M}{EI}dz + \int \overline{N}.\alpha.T_m.dz + \int \overline{M}.\alpha\frac{T_{inf}-T_{sup}}{h}.dz$$

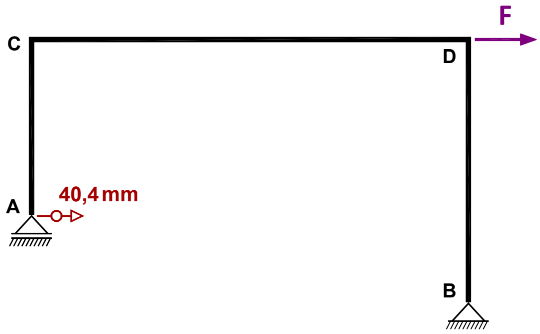
$$\theta_B + 0{,}2\times0{,}005 = 3{,}611\times10^{-6} - 4{,}71235\times10^{-3} - 1{,}8\times10^{-5} - 5{,}25\times10^{-3}$$

$$\theta_B = -1{,}098\times10^{-2}\ \text{rad} = -0{,}629^{o}$$

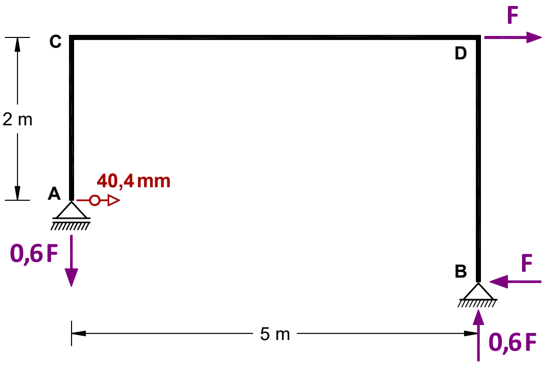
Alínea c)

Na **alínea a)** foi determinado o deslocamento horizontal do apoio **A**: $\delta_A^h = 4{,}04 \times 10^{-2}\,\mathrm{m} = 40{,}4\,\mathrm{mm}$ ⇐

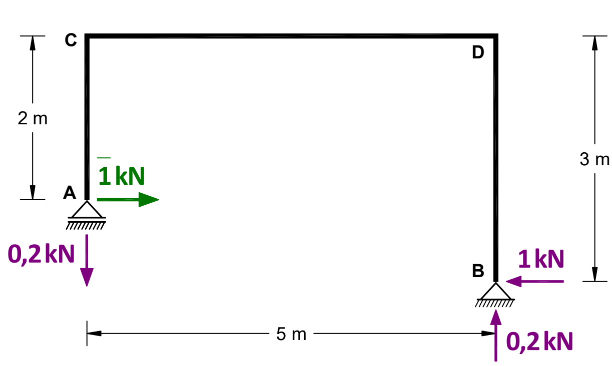
Pretende-se calcular a força horizontal a actuar em **D** que anule esse deslocamento. Ou seja, quando na estrutura actua uma força horizontal **F**, o deslocamento do ponto **A** deverá ser: $\delta_A^h = 40{,}4\,\mathrm{mm}$ ⇒



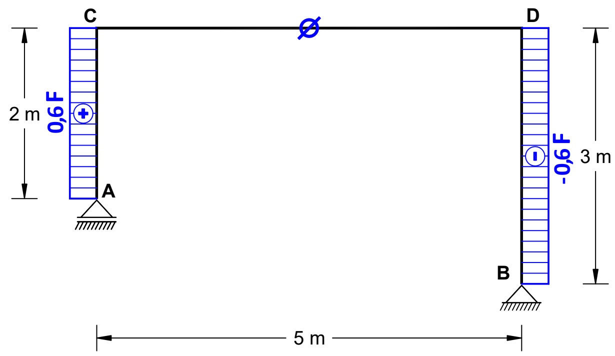
Sistema real



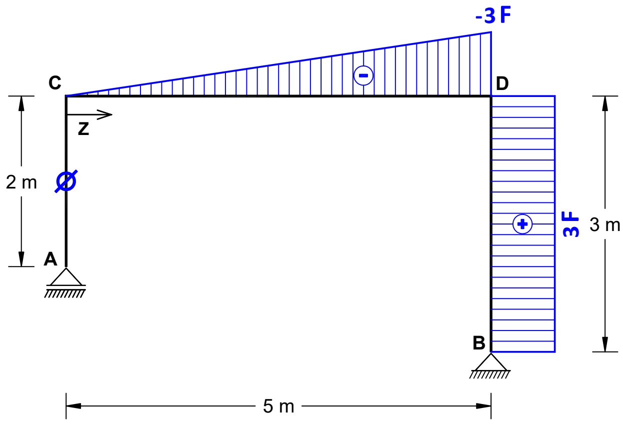
Sistema virtual



- **Diagramas de Esforços – sistema real**

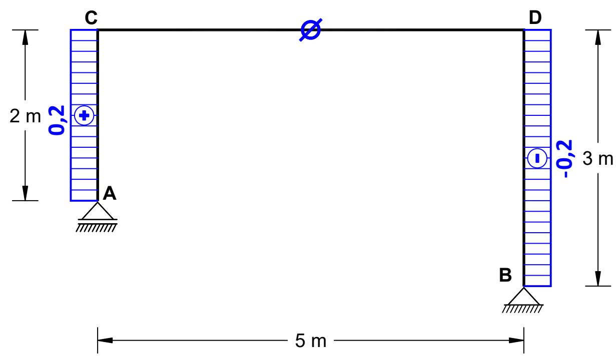


Esforço axial **N** (kN)

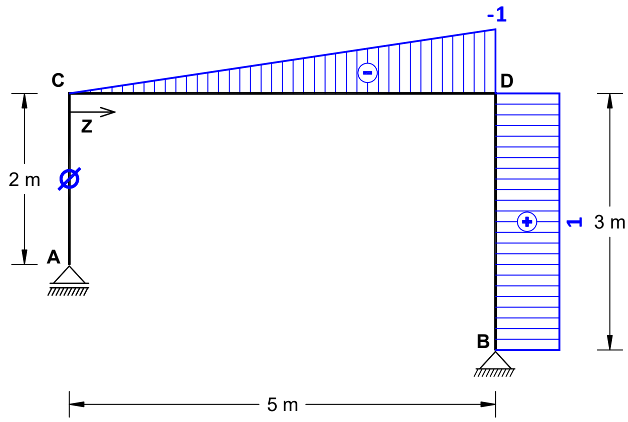


Momento flector **M** (kNm)

- **Diagramas de Esforços – sistema virtual**



Esforço axial $\overline{\mathbf{N}}$ (kN)



Momento flector $\overline{\mathbf{M}}$ (kNm)

TEOREMA DOS TRABALHOS VIRTUAIS (desprezando a contribuição do esforço transverso):

$$\boxed{\bar{1} \text{ x } \delta_A + \Sigma\left(\bar{R} \times \text{assent. apoio}\right) = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \int \bar{N} . \alpha . T_m . dz + \int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} . dz}$$

$$\int \frac{\bar{N} \times N}{EA} dz = \int_A^C \frac{\bar{N} \times N}{EA} dz + \int_B^D \frac{\bar{N} \times N}{EA} dz + \int_C^D \frac{\bar{N} \times N}{EA} dz =$$

$$= \frac{1}{3{,}6 \times 10^6} \left[0{,}2 \times 0{,}6 \text{ F} \times 2 + (-0{,}2) \times (-0{,}6 \text{ F}) \times 3\right] + 0 = \frac{F}{6} \times 10^{-6}$$

$$\int \frac{\bar{M} \times M}{EI} dz = \left[\int \frac{\bar{M} \times M}{EI} dz\right]_{Pilares} + \left[\int \frac{\bar{M} \times M}{EI} dz\right]_{Viga} = 1{,}875 \text{ F} \times 10^{-4} + \frac{20 \text{F}}{9{,}375} \times 10^{-4} = 4{,}0083 \text{ F} \times 10^{-4}$$

$$\left[\int \frac{\bar{M} \times M}{EI} dz\right]_{Pilares} = \frac{1}{4{,}8 \times 10^4} \times \frac{3 \times 3 \text{F} \times 3}{3} = \frac{3 \text{F}}{1{,}6 \times 10^4} = 1{,}875 \text{ F} \times 10^{-4}$$

$$\left[\int \frac{\bar{M} \times M}{EI} dz\right]_{Viga} = \frac{1}{9{,}375 \times 10^4} \times \frac{+3\text{F} \times 5}{6} \times (-2 - 2 \times 3) = \frac{20 \text{F}}{9{,}375} \times 10^{-4}$$

$$\bar{1} \text{ x } \delta_A^h + \Sigma\left(\bar{R} \times \text{assent. apoio}\right) = \int \frac{\bar{N} \times N}{EA} dz + \int \frac{\bar{M} \times M}{EI} dz + \int \bar{N} . \alpha . T_m . dz + \int \bar{M} . \alpha \frac{T_{inf} - T_{sup}}{h} . dz$$

$$1 \times 40{,}4 \times 10^{-3} = \frac{F}{6} \times 10^{-6} + 4{,}0083 \text{ F} \times 10^{-4}$$

$$F = 100{,}75 \text{ kN} \rightarrow$$