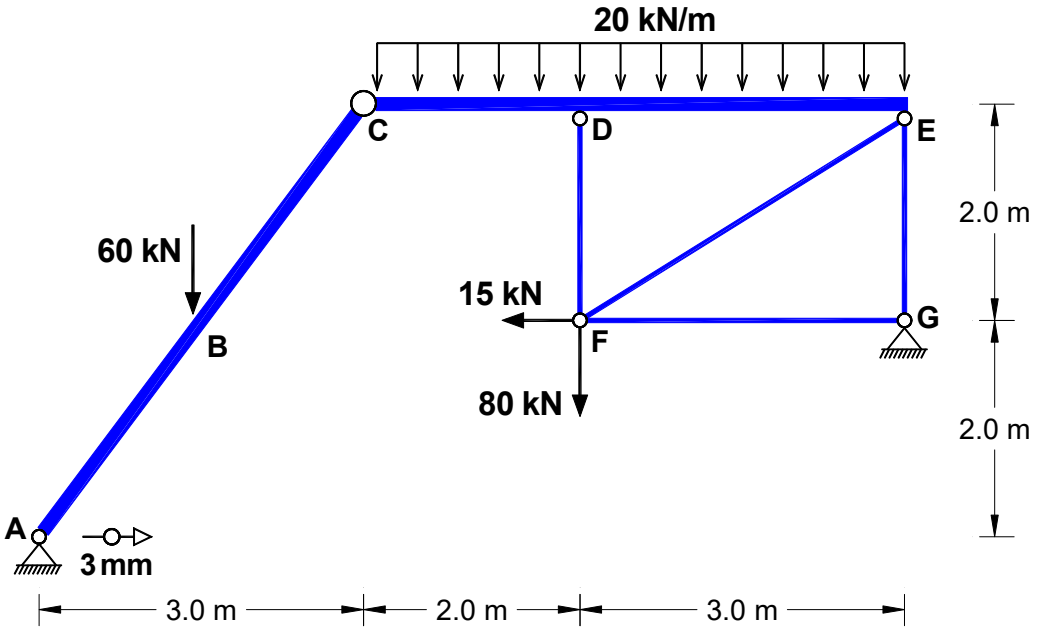
Instituto Superior de Engenharia do Porto

LICENCIATURA EM ENGENHARIA CIVIL

TEORIA DE ESTRUTURAS

TEOREMA DOS TRABALHOS VIRTUAIS



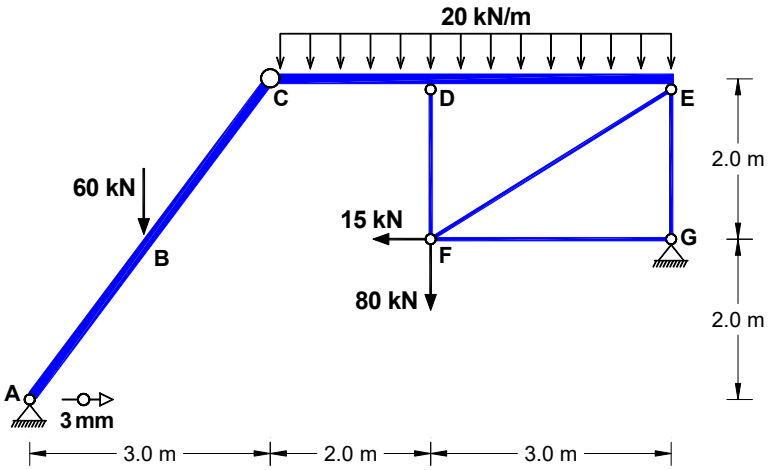
ESTRUTURA MISTA ISOSTÁTICA

FOLHA 5 – EX. 1

ISABEL ALVIM TELES

EXERCÍCIO

Considere a estrutura representada na figura.



Barras AC e **CE**

Secção: 0,25m x 0,30m

Betão: E = 30 GPa

Restantes barras

Perfil tubular: Ø = 60 mm

esp = 5 mm

Aço: E = 200 GPa

Responda às alíneas seguintes desprezando a contribuição do esforço transverso.

a) Determine a rotação relativa das secções à direita e à esquerda da rótula **C**;

b) Determine o deslocamento relativo dos pontos **A** e **E**.

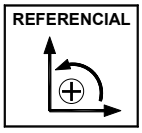
RESOLUÇÃO

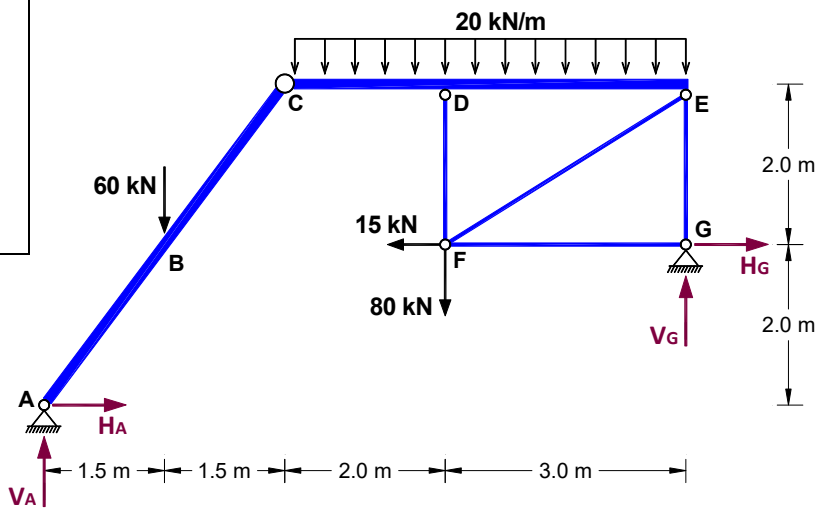
- **SISTEMA REAL** - **Reacções nos apoios**

Arco de 3 rótulas

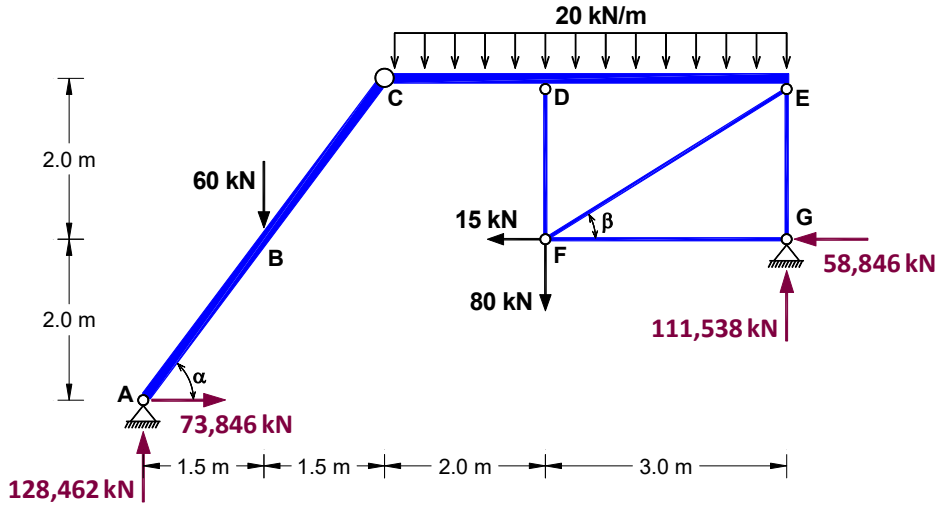
$$\text{Toda a estrutura} \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_G = 0 \end{cases}$$

$$\text{Corpo ABC} \Rightarrow \sum M_C = 0$$



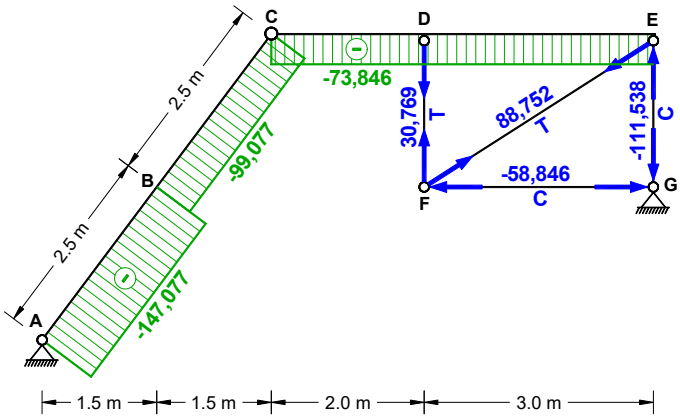


$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_G = 0 \\ \sum M_C^{ABC} = 0 \end{cases} \Rightarrow \begin{cases} H_A + H_G - 15 = 0 \\ V_A + V_G - 60 - 80 - 20 \times 5 = 0 \\ H_A \times 2 - V_A \times 8 + 60 \times 6{,}5 + 80 \times 3 + 100 \times 2{,}5 = 0 \\ H_A \times 4 - V_A \times 3 + 60 \times 1{,}5 = 0 \end{cases} \Rightarrow \begin{cases} H_A = 73{,}846 \text{ kN} \rightarrow \\ V_A = 128{,}462 \text{ kN} \uparrow \\ H_G = -58{,}846 \text{ kN} \leftarrow \\ V_G = 111{,}538 \text{ kN} \uparrow \end{cases}$$

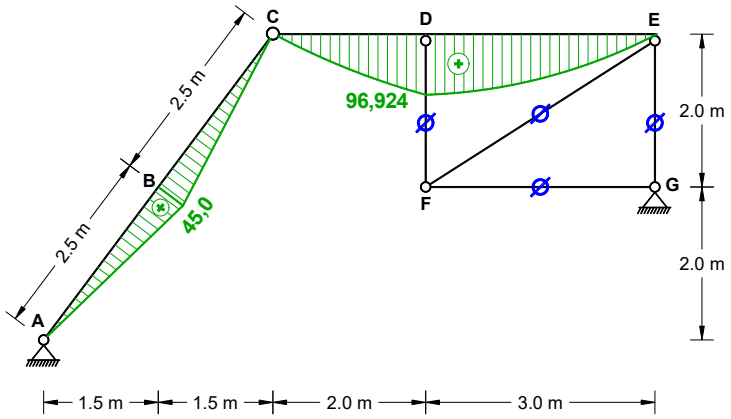


$$\alpha \Rightarrow \begin{cases} \cos\alpha = 0{,}8 \\ \text{sen}\,\alpha = 0{,}6 \end{cases} \qquad \beta = 33{,}69^\circ \Rightarrow \begin{cases} \cos\beta = 0{,}8321 \\ \text{sen}\,\beta = 0{,}5547 \end{cases}$$

SISTEMA REAL - Esforços instalados nas barras



Esforço axial **N** (kN)



Momento fletor **M** (kNm)

Tramo CD $M(z) = 68{,}462\ z - 10\,z^2$

Tramo DE $M(z) = 96{,}924 - 2{,}307\ z - 10\,z^2$

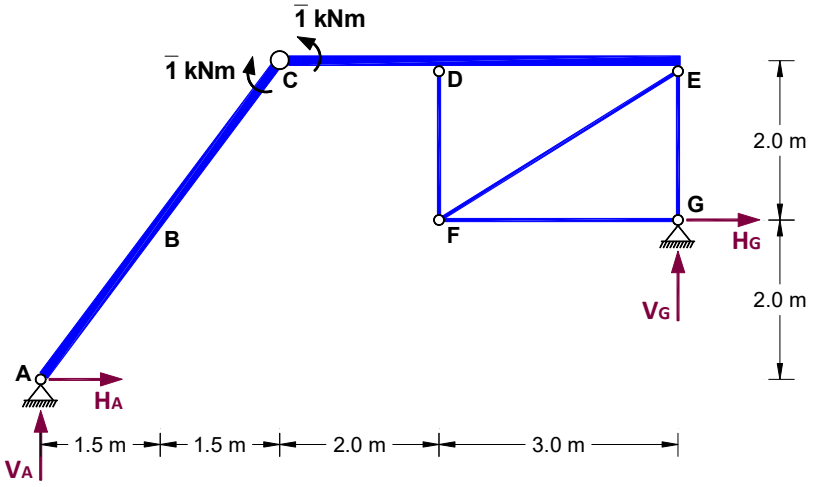
a) Rotação relativa das secções à esquerda e à direita da rótula C

- **Sistema virtual - Reacções nos apoios**

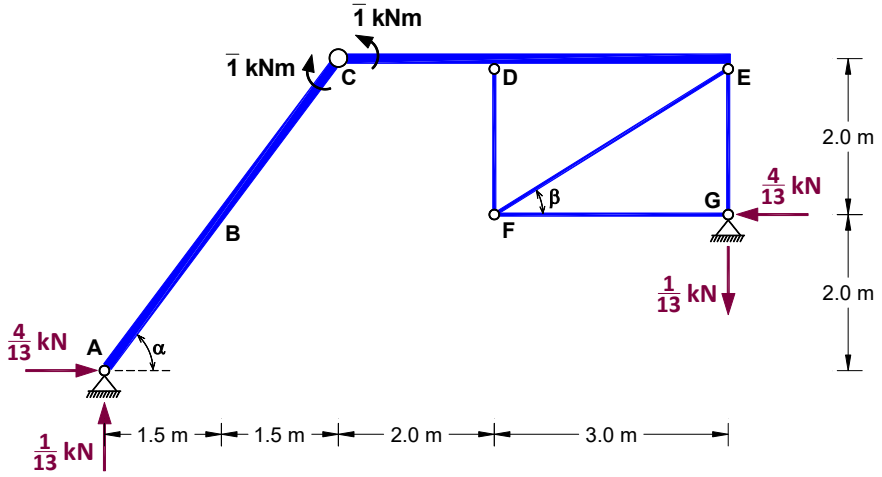
Arco de 3 rótulas

$$\text{Toda a estrutura} \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_G = 0 \end{cases} \qquad \text{Corpo ABC} \Rightarrow \sum M_C = 0$$

REFERENCIAL

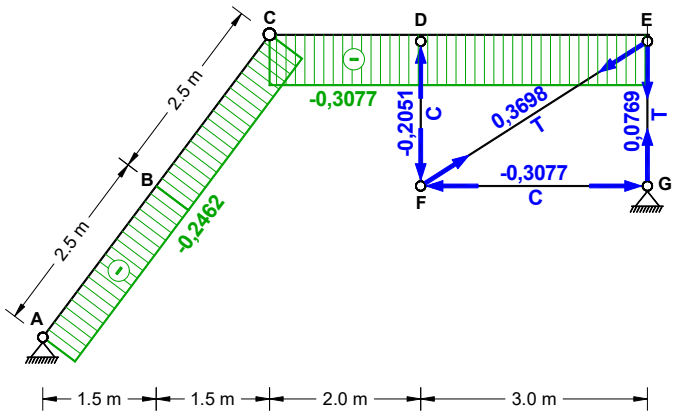


$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_G = 0 \\ \sum M_C^{ABC} = 0 \end{cases} \Rightarrow \begin{cases} H_A + H_G = 0 \\ V_A + V_G = 0 \\ H_A \times 2 - V_A \times 8 + 1 - 1 = 0 \\ H_A \times 4 - V_A \times 3 - 1 = 0 \end{cases} \Rightarrow \begin{cases} H_A = \frac{4}{13}\ \text{kN} \rightarrow \\ V_A = \frac{1}{13}\ \text{kN} \uparrow \\ H_G = -\frac{4}{13}\ \text{kN} \leftarrow \\ V_G = -\frac{1}{13}\ \text{kN} \downarrow \end{cases}$$

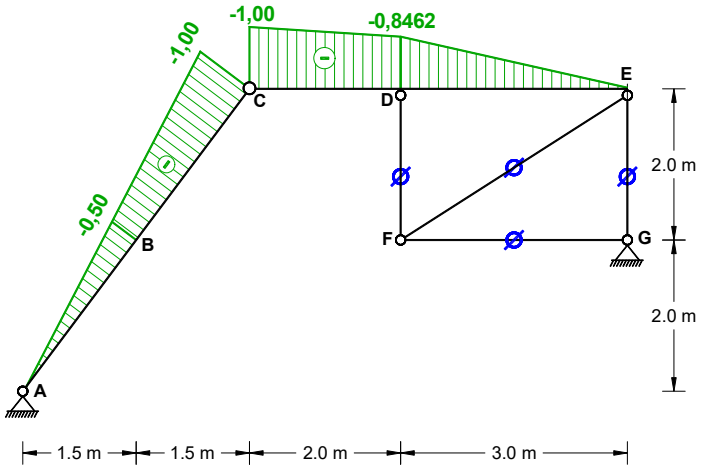


$$\alpha \Rightarrow \begin{cases} \cos\alpha = 0{,}8 \\ \text{sen}\,\alpha = 0{,}6 \end{cases} \qquad \beta = 33{,}69^\circ \Rightarrow \begin{cases} \cos\beta = 0{,}8321 \\ \text{sen}\,\beta = 0{,}5547 \end{cases}$$

SISTEMA VIRTUAL - Esforços instalados nas barras



Esforço axial **N** (kN)

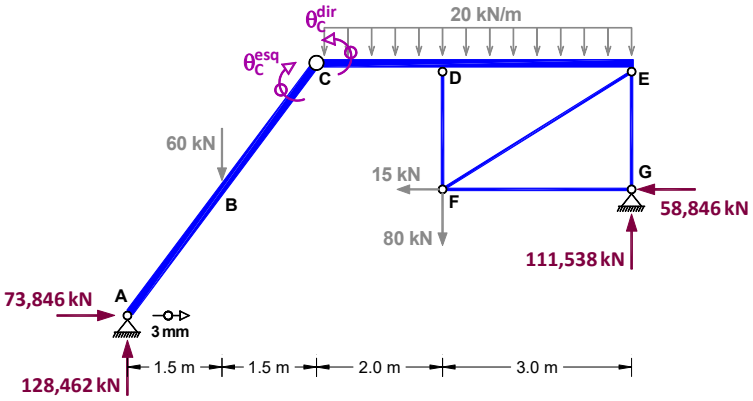


Momento fletor **M** (kNm)

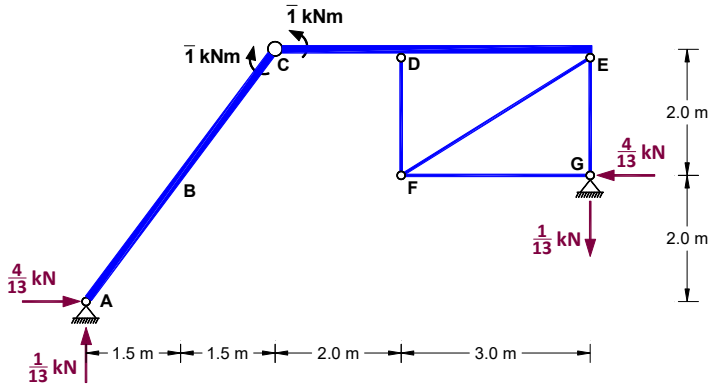
Tramo CD $M(z) = -1 + \frac{1}{13}\,\mathbf{z}$ kNm

Tramo DE $M(z) = -0{,}8462 + 0{,}2820\,\mathbf{z}$ kNm

- **Aplicação do Teorema dos Trabalhos Virtuais (TTV)**



S - Sistema real



$\overline{\mathbf{S}}$ - Sistema virtual

$$\overline{1} \times \theta_C^{esq} + \overline{1} \times \theta_C^{dir} + \sum \bar{\mathbf{R}} \times \text{assent. apoio} = \underbrace{\int \frac{\bar{N} N}{E\,A}\,dz + \int \frac{\bar{M}\,M}{E\,I}\,dz}_{\text{barras contínuas}} + \underbrace{\sum \bar{N}\frac{N\,L}{E\,A}}_{\text{barras bi-articuladas}}$$

Barras contínuas
- $A = 0{,}075\ m^2$
- $I = 5{,}625 \times 10^{-4}\ m^4$
- $E = 30 \times 10^6\ kPa$
- $E\,A = 2{,}25 \times 10^6\ kPa.m^2$
- $E\,I = 16875\ kPa.m^4$

Barras bi-articuladas
- $A = \pi \times 2{,}75 \times 10^{-4}\ m^2$
- $E = 200 \times 10^6\ kPa$
- $E\,A = 55000\,\pi\ kPa.m^2$

Barra AB

$$\int_L \frac{\overline{N}\,N}{E\,A}dz = \frac{1}{2{,}25x10^6}\int_0^{2,5}\overline{N}\times N\ dz = \frac{1}{2{,}25x10^6}\times(-0{,}2462)\times(-147{,}077)\times 2{,}5 = 40{,}234\times10^{-6}$$

$$\int_L \frac{\overline{M}\,M}{E\,I}dz = \frac{1}{16875}\int_0^{2,5}\overline{M}\times M\ dz = \frac{1}{16875}\times\frac{1}{3}\times(-0{,}5)\times 45\times 2{,}5 = -1{,}111\times10^{-3}$$

Barra BC

$$\int_L \frac{\overline{N}\,N}{E\,A}dz = \frac{1}{2{,}25x10^6}\int_0^{2,5}\overline{N}\times N\ dz = \frac{1}{2{,}25x10^6}\times(-0{,}2462)\times(-99{,}077)\times 2{,}5 = 27{,}103\times10^{-6}$$

$$\int_L \frac{\overline{M}\,M}{E\,I}dz = \frac{1}{16875}\int_0^{2,5}\overline{M}\times M\ dz = \frac{1}{16875}\times\frac{45\times 2{,}5}{6}\times(2\times(-0{,}5)-1) = -2{,}222\times10^{-3}$$

Barra CD

$$\int_L \frac{\overline{N}\,N}{E\,A}dz = \frac{1}{2{,}25x10^6}\int_0^{2}\overline{N}\times N\ dz = \frac{1}{2{,}25x10^6}\times(-0{,}3077)\times(-73{,}846)\times 2 = 20{,}198\times10^{-6}$$

$$\int_L \frac{\overline{M}\,M}{E\,I}dz = \frac{1}{16875}\int_0^{2}\overline{M}\times M\ dz = \frac{1}{16875}\int_0^{2}\left(\left(-1+\frac{1}{3}\mathbf{Z}\right)\times\left(68{,}462\,\mathbf{Z}-10\,\mathbf{Z}^2\right)\right)dz =$$

$$= \frac{1}{16875}\times(-99{,}291) = -5{,}884\times10^{-3}$$

Barra DE

$$\int_L \frac{\overline{N}\,N}{E\,A}dz = \frac{1}{2{,}25x10^6}\int_0^{3}\overline{N}\times N\ dz = \frac{1}{2{,}25x10^6}\times(-0{,}3077)\times(-73{,}846)\times 3 = 30{,}297\times10^{-6}$$

$$\int_L \frac{\overline{M}\,M}{E\,I}dz = \frac{1}{16875}\int_0^{3}\overline{M}\times M\ dz = \frac{1}{16875}\int_0^{2}\left((-0{,}8462+0{,}2820\,\mathbf{Z})\times\left(96{,}924-2{,}307\,\mathbf{Z}-10\,\mathbf{Z}^2\right)\right)dz =$$

$$= \frac{1}{16875}\times(-101{,}072) = -5{,}989\times10^{-3}$$

Barras bi-articuladas

BARRAS	**L** (m)	**E A** (kPa.m^2)	**N** (kN)	$\overline{\mathbf{N}}$ (kN)	$\overline{N}\frac{N\,L}{E\,A}$
FG	3	55000 π	-58,846	-0,3077	31,438 x 10^{-5}
DG	2	55000 π	30,769	-0,2051	- 7,305 x 10^{-5}
EG	2	55000 π	-111,538	0,0769	- 9,928 x 10^{-5}
FE	$\sqrt{13}$	55000 π	88,752	0,3698	68,486 x 10^{-5}
Σ					82,692 x 10^{-5}

$$\sum \bar{R} \times \text{assent. apoio} = \frac{4}{13} \times 0{,}003 = \frac{0{,}012}{13}$$

Rotação relativa da secção à esquerda e à direita de C:

$$\bar{1} \times \theta_C^{esq} + \bar{1} \times \theta_C^{dir} + \sum \bar{R} \times \text{assent. apoio} = \int \frac{\bar{N} N}{E A} dz + \int \frac{\bar{M} M}{E I} dz + \sum \bar{N} \frac{N L}{E A}$$

$$\Delta\theta_C^{esq/dir} = \theta_C^{esq} + \theta_C^{dir} =$$

$$= -\frac{0{,}012}{13} + (40{,}234 + 27{,}103 + 20{,}198 + 30{,}297) \times 10^{-6} + (-1{,}111 - 2{,}222 - 5{,}884 - 5{,}989) \times 10^{-3} + 82{,}692 \times 10^{-5}$$

$$\Delta\theta_C^{esq/dir} = \theta_C^{esq} + \theta_C^{dir} = -\frac{0{,}012}{13} + 117{,}832 \times 10^{-6} - 15{,}206 \times 10^{-3} + 82{,}692 \times 10^{-5} = -1{,}518 \times 10^{-2} \text{ rad}$$

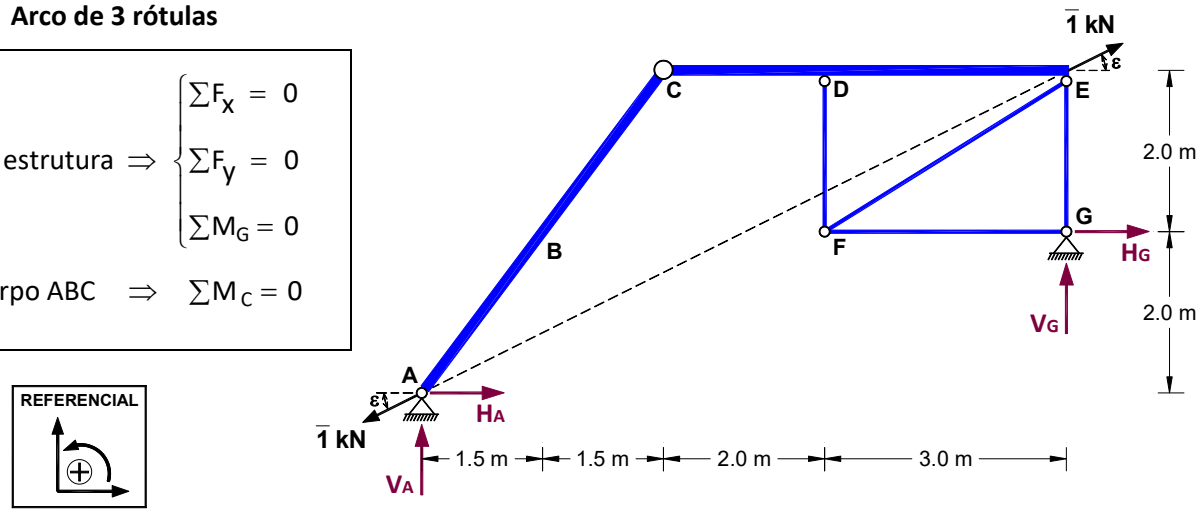
a) Deslocamento relativo dos pontos A e E

- **Sistema virtual - Reacções nos apoios**

Arco de 3 rótulas

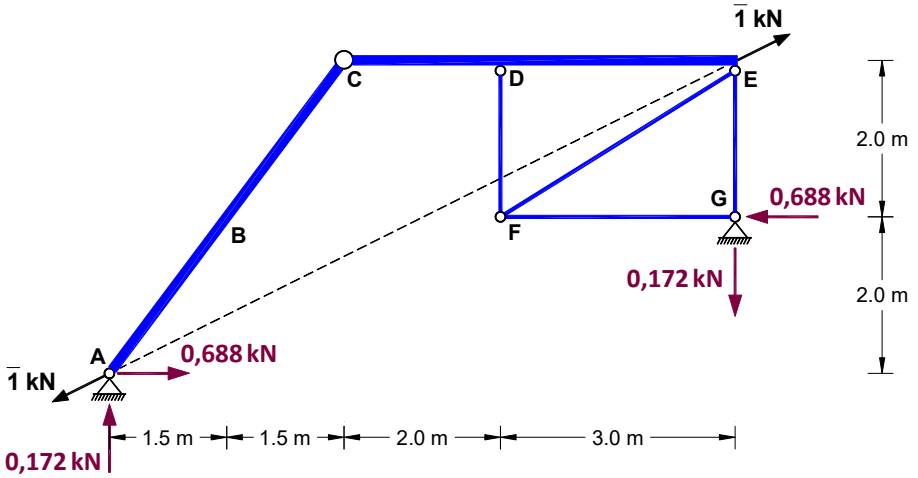
$$\text{Toda a estrutura} \Rightarrow \begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_G = 0 \end{cases}$$

$$\text{Corpo ABC} \Rightarrow \sum M_C = 0$$

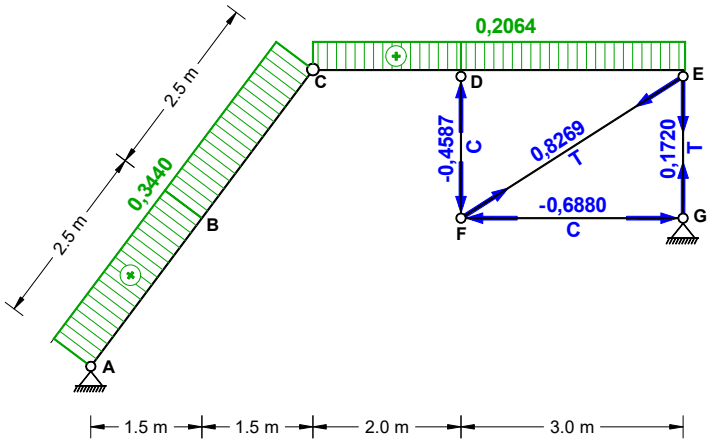


$$\varepsilon = tg^{-1}\left(\frac{4}{8}\right) = 26{,}565^\circ \Rightarrow \begin{cases} \cos\varepsilon = 0{,}8944 \\ \text{sen}\,\varepsilon = 0{,}4472 \end{cases}$$

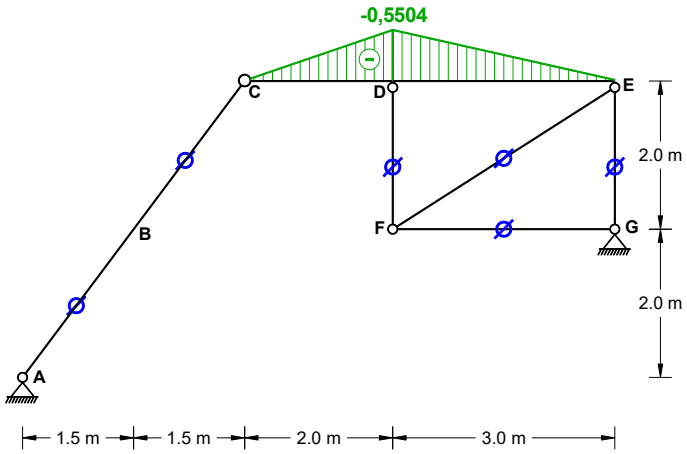
$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_G = 0 \\ \sum M_C^{ABC} = 0 \end{cases} \Rightarrow \begin{cases} H_A + H_G = 0 \\ V_A + V_G = 0 \\ H_A \times 2 - V_A \times 8 = 0 \\ H_A \times 4 - V_A \times 3 - 4 \times \cos\varepsilon + 3 \times \text{sen}\,\varepsilon = 0 \end{cases} \Rightarrow \begin{cases} H_A = 0{,}684 \text{ kN} \rightarrow \\ V_A = 0{,}172 \text{ kN} \uparrow \\ H_G = -0{,}688 \text{ kN} \leftarrow \\ V_G = -0{,}172 \text{ kN} \downarrow \end{cases}$$



SISTEMA VIRTUAL - Esforços instalados nas barras



Esforço axial **N** (kN)

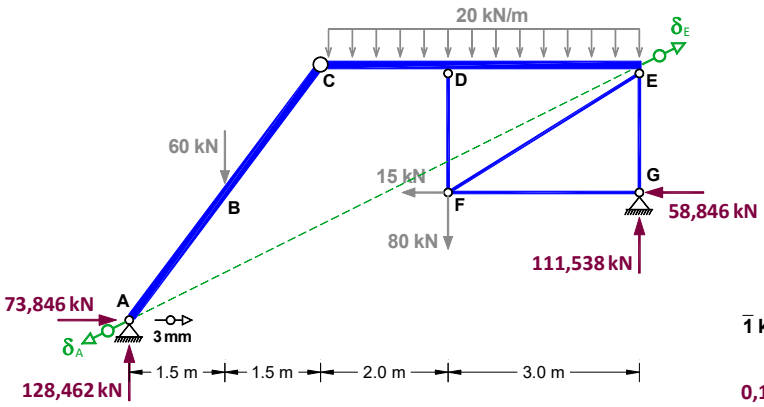


Momento fletor **M** (kNm)

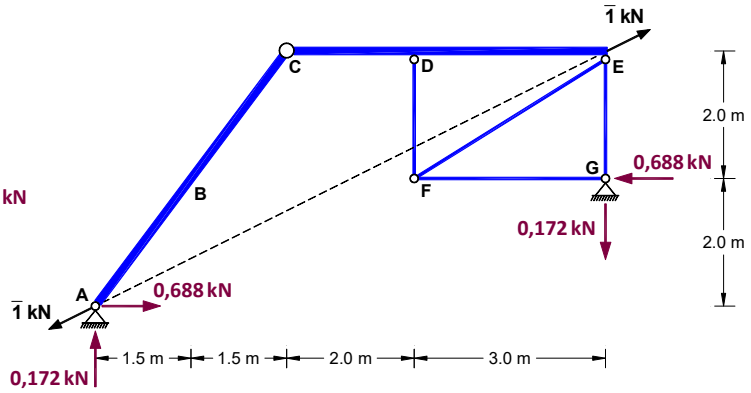
Tramo CD $M(z) = -0{,}2752\,\mathbf{z}$ kNm

Tramo DE $M(z) = -0{,}5504 + 0{,}1835\,\mathbf{z}$ kNm

- **Aplicação do Teorema dos Trabalhos Virtuais (TTV)**



S - Sistema real



$\overline{\mathbf{S}}$ - Sistema virtual

$$\bar{1} \times \delta_A + \bar{1} \times \delta_E + \sum \bar{\mathbf{R}} \times \text{assent. apoio} = \underbrace{\int \frac{\bar{N} N}{E A}\, dz + \int \frac{\bar{M} M}{E I}\, dz}_{\text{barras contínuas}} + \underbrace{\sum \bar{N} \frac{N L}{E A}}_{\text{barras bi-articuladas}}$$

Barras contínuas $\begin{cases} A = 0{,}075\ m^2 \\ I = 5{,}625 \times 10^{-4}\ m^4 \\ E = 30 \times 10^6\ kPa \\ E A = 2{,}25 \times 10^6\ kPa.m^2 \\ E I = 16875\ kPa.m^4 \end{cases}$

Barras bi-articuladas $\begin{cases} A = \pi \times 2{,}75 \times 10^{-4}\ m^2 \\ E = 200 \times 10^6\ kPa \\ E A = 55000\,\pi\ kPa.m^2 \end{cases}$

Barra AB

$$\int_L \frac{\bar{N} N}{E A} dz = \frac{1}{2{,}25 \times 10^6} \int_0^{2,5} \bar{N} \times N\ dz = \frac{1}{2{,}25 \times 10^6} \times 0{,}344 \times (-147{,}077) \times 2{,}5 = -56{,}216 \times 10^{-6}$$

$$\int_L \frac{\bar{M} M}{E I} dz = 0$$

Barra BC

$$\int_L \frac{\bar{N} N}{E A} dz = \frac{1}{2{,}25 \times 10^6} \int_0^{2,5} \bar{N} \times N\ dz = \frac{1}{2{,}25 \times 10^6} \times 0{,}344 \times (-99{,}077) \times 2{,}5 = -37{,}869 \times 10^{-6}$$

$$\int_L \frac{\bar{M} M}{E I} dz = 0$$

Barra CD

$$\int_L \frac{\bar{N} N}{E A} dz = \frac{1}{2{,}25 \times 10^6} \int_0^{2} \bar{N} \times N\ dz = \frac{1}{2{,}25 \times 10^6} \times 0{,}2064 \times (-73{,}846) \times 2 = -13{,}548 \times 10^{-6}$$

$$\int_L \frac{\bar{M} M}{E I} dz = \frac{1}{16875} \int_0^{2} \bar{M} \times M\ dz = \frac{1}{16875} \int_0^{2} \left((-0{,}2752\ \mathbf{Z}) \times \left(68{,}462\ \mathbf{Z} - 10\ \mathbf{Z}^2 \right) \right) dz =$$

$$= \frac{1}{16875} \times (-39{,}234) = -2{,}325 \times 10^{-3}$$

Barra DE

$$\int_L \frac{\bar{N} N}{E A} dz = \frac{1}{2{,}25 \times 10^6} \int_0^{3} \bar{N} \times N\ dz = \frac{1}{2{,}25 \times 10^6} \times 0{,}2064 \times (-73{,}846) \times 3 = -20{,}322 \times 10^{-6}$$

$$\int_L \frac{\bar{M} M}{E I} dz = \frac{1}{16875} \int_0^{3} \bar{M} \times M\ dz = \frac{1}{16875} \int_0^{2} \left((-0{,}5504 + 0{,}1835\ \mathbf{Z}) \times \left(96{,}924 - 2{,}307\ \mathbf{Z} - 10\ \mathbf{Z}^2 \right) \right) dz =$$

$$= \frac{1}{16875} \times (-65{,}725) = -3{,}895 \times 10^{-3}$$

Barras bi-articuladas

BARRAS	L (m)	E A (kPa.m^2)	N (kN)	$\bar{N}$ (kN)	$\bar{N}\frac{N\,L}{E\,A}$
FG	3	55000 π	-58,846	-0,6880	70,293 x 10^{-5}
DG	2	55000 π	30,769	-0,4587	- 16,337 x 10^{-5}
EG	2	55000 π	-111,538	0,1720	- 22,206 x 10^{-5}
FE	$\sqrt{13}$	55000 π	88,752	0,8269	153,141 x 10^{-5}
Σ					184,891 x 10^{-5}

$\sum \bar{R}$ x assent. apoio = 0,688 x 0,003 = 2,064 x 10^{-3}

Deslocamento relativo dos pontos A e E:

$$\bar{1} \times \delta_A + \bar{1} \times \delta_E + \sum \bar{R} \text{ x assent. apoio} = \int \frac{\bar{N} N}{E\,A}\,dz + \int \frac{\bar{M}\,M}{E\,I}\,dz + \sum \bar{N}\frac{N\,L}{E\,A}$$

$\delta_A + \delta_E$ = -2,064 x 10^{-3} + (-56,216 - 37,869 - 13,548 - 20,322) x 10^{-6} + (-2,325 - 3,895) x 10^{-3}+ 184,891 x 10^{-5}

$\delta_A + \delta_E$ = -2,064 x 10^{-3} - 127,955 x 10^{-6} - 6,220 x 10^{-3}+ 184,891 x 10^{-5}= -6,563 x10^{-3} m

$\delta_A + \delta_E$ = -6,563 mm (o ponto **A** e o ponto **E** vão aproximar-se)